

OBSTRUCTIONS TO DEFORMING A SPACE CURVE

BY

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ABSTRACT. Mumford described a curve, γ , in $\mathbf{P}^3$ that has obstructed infinitesimal deformations (in fact the Hilbert scheme of the curve is generically nonreduced). This paper studies γ 's Hilbert scheme by studying deformations of γ in $\mathbf{P}^3$ over parameter spaces of the form $\text{Spec}(k[t]/(t^n))$, $n = 2, 3, \dots$. Given a deformation of γ over $\text{Spec}(k[t]/(t^n))$ one attempts to extend it to a deformation of γ over $\text{Spec}(k[t]/(t^{n+1}))$. If it will not extend, this deformation is said to be *obstructed at the n th order*.

I show that on a generic version of Mumford's curve, an infinitesimal deformation (i.e., a deformation over $\text{Spec}(k[t]/(t^2))$) is either obstructed at the second order, or at no order, in which case we say it is *unobstructed*.

Let F be a nonsingular cubic surface in $\mathbf{P}^3$, let E be one of the 27 lines on F , and let H be the hyperplane divisor on F . Mumford [3] showed that any nonsingular member γ of the linear system $|4H + 2E|$ represents a point of the Hilbert scheme $\mathbf{H}$ parametrizing curves of degree 14 and genus 24 at which $\mathbf{H}$ is not reduced. This can be restated in the language of infinitesimal deformations. An *infinitesimal deformation* of γ is a flat family of projective space curves over $\text{Spec}(k[t]/(t^2))$ with γ as special fiber. One attempts to lift such a deformation successively to deformations over $\text{Spec}(k[t]/(t^n))$, $n = 3, 4, \dots$. If there is an n such that the deformation extends modulo t^n , but not modulo t^{n+1} , we say the deformation is *obstructed at the n th order*. The fact that the Hilbert scheme is nonreduced can be interpreted as saying that γ has infinitesimal deformations that are obstructed at some order. More specifically, Mumford showed that the dimension of $\mathbf{H}$ at the point z_γ representing γ is 56 while the dimension of the tangent space to $\mathbf{H}$ at z_γ is 57. This guarantees the existence of infinitesimal deformations that are obstructed.

I will show that for a nonsingular element γ of $|4H + 2E|$ any infinitesimal deformation either is obstructed at the second order, or at no order (in which case the deformation is said to be *unobstructed*).

In §1, the generators and relations of the ideal defining a member of $|4H + 2E|$ are given. From these come a free resolution of the ideal, and a resolution of the structure sheaf of the nonsingular members. In §2, the deformations of γ in $\mathbf{P}^3$ are related to the homogeneous deformations of certain affine cones over γ . In §3, the unobstructed deformations of γ are obtained from the 2×2 minors of a 2×4 matrix of polynomials. In §4 and §5, the obstructed deformations are identified

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and the cohomology classes representing the second order obstructions are found. Finally (Theorem 2), these classes are seen to actually obstruct the deformations at the second order, at least on a general γ .

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Conventions. k is an algebraically closed field, $P = k[x_0, \dots, x_n]$, $\underline{x} = x_0, \dots, x_n$ (usually $n = 3$ and we use $x_0 = x, x_1 = y, x_2 = z, x_3 = w$), P_s = forms of degree s in P . $P(i)$ is defined by $P(i)_s = P_{i+s}$.

1. Mumford's curves. Let F be a nonsingular cubic surface in $\mathbf{P}^3$, H the hyperplane divisor of F , E any one of the 27 lines on F . Mumford's curves, γ , are the nonsingular members of the linear system $|4H + 2E|$.

Wahl [7, §3.6] has proven the following:

PROPOSITION 1.1. *Any of Mumford's curves is defined by a homogeneous ideal $I = (F, S_1, S_2, S_3)$, where the S_i are nonsingular sextic forms.*

Actually the proof of the proposition shows that any member of $|4H + 2E|$ is given by an ideal $I = (F, S_1, S_2, S_3)$ where the S_i are sextic forms but not necessarily nonsingular. We will give the S_i quite explicitly.

Let $V(f_1, \dots, f_n)$ be the variety in $\mathbf{P}^3$ defined by the homogeneous polynomials $f_1, \dots, f_n$.

Assume $E = V(x, y)$, changing co-ordinates if necessary. Then $F = xA + yB$, A and B quadratic forms. Since F is nonsingular, A and B have no common component modulo the ideal (x, y) . In fact, if $L|A$ and $L|B \pmod{(x, y)}$, the Jacobian criterion shows that $V(x, y, L)$ are singular points of F .

PROPOSITION 1.2. *$D \in |4H + 2E|$ if and only if D is defined by an ideal of the form $I(\eta, \xi) = (F, x^2\eta + B^2\xi, xy\eta - AB\xi, y^2\eta + A^2\xi)$, η a quartic form, ξ a quadratic form.*

PROOF. We will show that ideals of the form $I(\eta, \xi)$ define elements of $|4H + 2E|$, then show, by comparing dimensions, that divisors defined by such ideals are all of $|4H + 2E|$.

For $\eta \notin (F)$, it is clear that $I(\eta, 0)$ defines an element of $|4H + 2E|$.

If $\xi \neq 0$, let $g_1 = x, g_2 = y, g_3 = A, g_4 = B$. Let $W_i = \mathbf{P}^3 \sim V(g_i)$, and $U_i = W_i \cap F$. Let D be given by $I(\eta, \xi)$.

On U_1 , the Cartier divisor D is defined by $f_1 = S_1/x^6$; on U_2 by $f_2 = S_3/y^6$; on U_3 by $f_3 = S_3/A^3$; on U_4 by $f_4 = S_1/B^3$.

We claim that D is linearly equivalent to D' defined by $I(q, 0)$, $q \notin (F)$. In fact, D' is defined on U_1 by $f'_1 = S'_1/x^6 = x^2q/x^6$; etc. It is easy to check that $f_i/f'_i = S_2/S'_2$, for all i . For example, on U_1 , the quotient

$$\begin{aligned} S_2/S'_2 &= (xy\eta - AB\xi)/xyq = (x^2y\eta - xAB\xi)/x^2yq \\ &= (x^2y\eta + yB^2\xi)/x^2yq = (S_1/x^6) \cdot (x^6/S'_1). \end{aligned}$$

Thus, for all i , $f_i/f'_i = f_1/f'_1$ in K , and $D \equiv D'$. Hence $D \in |4H + 2E|$.

Now we have a map $\text{Proj}(P_4 \oplus P_2) \xrightarrow{\phi} |4H + 2E|$ given by $\phi([\eta, \xi]) =$ (the divisor defined by $I(\eta, \xi)$). $\text{Dim}|4H + 2E| = 37$, $\text{Dim}(P_4 \oplus P_2) = 35 + 10 = 45$;

thus $\dim(\text{Proj}(P_4 \oplus P_2)) = 44$. It is not hard to see that the kernel of ϕ is 7-dimensional, spanned by the 4-dimensional family $(\eta, 0)$, $\eta \in (F)$, and the 3-dimensional family $(0, \xi) - (aB^2 - bAB + cA^2, 0)$, where $\xi = ax^2 + bxy + cy^2$. Thus $\dim(\text{im } \phi) = 44 - 7 = 37$, so ϕ is onto.

PROPOSITION 1.3. *The relation module of $I(\eta, \xi)$ is generated by $r_1 = (-B\xi, y, -x, 0)$, $r_2 = (A\xi, 0, y, -x)$, $r_3 = (-x\eta, A, B, 0)$, $r_4 = (-y\eta, 0, A, B)$.*

PROOF. We claim the following sequence is exact:

$$0 \rightarrow P(-9) \xrightarrow{\phi_3} P(-7)^2 \oplus P(-8)^2 \xrightarrow{\phi_2} P(-3) \oplus P(-6)^3 \xrightarrow{\phi_1} I \rightarrow 0,$$

where $\phi_1 = (F, S_1, S_2, S_3)$,

$$\phi_2 = \begin{bmatrix} -B\xi & A\xi & -x\eta & -y\eta \\ y & 0 & A & 0 \\ -x & y & B & A \\ 0 & -x & 0 & B \end{bmatrix},$$

$\phi_3 = (A, B, y, -x)$. The exactness follows easily from [1, Corollary 2]. Then we observe that the relation module is generated by the columns of ϕ_2 .

Taking associated sheaves in the exact sequence above we obtain:

PROPOSITION 1.4. $\mathcal{O}_Y$ has a resolution

$$0 \rightarrow \mathcal{O}(-9) \rightarrow \mathcal{O}(-7)^2 \oplus \mathcal{O}(-8)^2 \rightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-6)^3 \rightarrow \mathcal{O} \rightarrow \mathcal{O}_Y \rightarrow 0$$

as an $\mathcal{O} = \mathcal{O}_{\mathbf{P}^3}$ -module.

2. Deformations of cones. Consider the Hilbert scheme $\mathbf{H}$ parametrizing subvarieties of $\mathbf{P}^n$ having the same Hilbert polynomial as a given subvariety X . In this section we will see that $\mathbf{H}$ is locally the same as the homogeneous deformations of the cones over X defined by certain homogeneous ideals (Theorem 1).

As in [4], we let $\mathcal{C}$ be the category of Artinian local K -algebras with residue field k and $\hat{\mathcal{C}}$ be the category of complete Noetherian local k -algebras A for which $A/\mathfrak{m}^n$ is in $\mathcal{C}$ for all n . A functor on $\mathcal{C}$ is prorepresentable if it is represented by an object of $\hat{\mathcal{C}}$.

Let X be a subscheme of $\mathbf{P}^n$. Let $I = (f_1(\underline{x}), \dots, f_m(\underline{x}))$, $f_i(\underline{x})$ homogeneous of degree n_i , be an ideal defining X . Define $V(I) \subset A^{n+1}$ to be the cone over X defined by I . Note that $V(I)$ is different for different I defining X .

DEFINITION 1. Let $T \in \mathcal{C}$ be given, its maximal ideal M_T . A deformation of the cone $V(I)$ over $\text{Spec } T$ is a closed subscheme V of $A^{n+1} \times_{\text{Spec } k} \text{Spec } T$, that is flat over $\text{Spec } T$, and is defined by m equations $F_i \in T[\underline{x}]$, homogeneous of degree n_i in $\underline{x}$, such that $F_i = f_i(\underline{x}) \bmod (M_T T[\underline{x}])$.

V is flat over $\text{Spec } T$ if and only if it satisfies the *lifting of relations* condition: every relation $(r_1, \dots, r_m)$ on $(f_1, \dots, f_m)$, i.e. $\sum_{i=1}^m r_i f_i = 0$, lifts to a relation $(R_1, \dots, R_m)$ on $(F_1, \dots, F_m)$, i.e. $\sum R_i F_i = 0$ and $R_i = r_i(\underline{x}) \bmod (M_T T[\underline{x}])$.

Let $F(T)$ be the set of deformations of the cone $V(I)$ over $\text{Spec } T$, $T \in \mathcal{C}$. Suppose we are given a map in $\mathcal{C}$, $\phi: S \rightarrow T$, and $W \in F(S)$. W is defined by $F_i \in S[\underline{x}]$. Now ϕ induces a natural map $\phi: S[\underline{x}] \rightarrow T[\underline{x}]$. Let $F(\phi)(W) \in F(T)$ be

given by $F'_i = \phi(F_i)$. (Flatness is easy to check using the lifting of relations $R'_j = \phi(R_j)$.) Thus $F(\cdot)$ is a covariant functor on C .

PROPOSITION 2.1. $F(\cdot)$ is prorepresentable.

PROOF. We will use Theorem 2.11 [4]. $F(k)$ is a one point set, namely $\{V(I)\}$. Given $A' \rightarrow A$ and $A'' \rightarrow A$ in C , it is easy to see that $(A' \times_A A'')[\underline{x}] \rightarrow A'[\underline{x}] \times_{A[\underline{x}]} A''[\underline{x}]$ is an isomorphism. It follows that $F(A' \times_A A'') \rightarrow F(A') \times_{F(A)} F(A'')$ is an isomorphism. This establishes (H_1) , (H_2) , and (H_4) . The tangent space $t_F = F(k[\varepsilon]) = \text{Hom}_P(I, P/I)_0$ is finite dimensional, so (H_3) holds. Then $F(\cdot)$ is prorepresentable.

Thus $F(\cdot) = h_A(\cdot)$, for some $A \in \hat{C}$.

Recall that t_F is the Zariski tangent space to $\text{Spec } A$, the space of infinitesimal deformations of $V(I)$.

Consider the algebras $k[t]/(t^n)$, $n = 2, 3, \dots$. Given an element V of $F(k[t]/(t^n))$ can it be extended to an element V' of $F(k[t]/(t^{n+1}))$? That is, does there exist a V' such that $F(\phi)(V') = V$, where ϕ is the natural map $k[t]/(t^{n+1}) \rightarrow k[t]/(t^n)$. Thus we have equations $f_i(\underline{x}; t) \bmod(t^n)$ for V , so that for any relation $r_i(\underline{x}; t)$, $\sum f_i(\underline{x}; t)r_i(\underline{x}; t) = 0 \bmod(t^n)$. To get V' , we need Δf_i and Δr_i , homogeneous of the same degree in $\underline{x}$ as f_i and r_i , so that

$$\sum (f_i + t^n \Delta f_i)(r_i + t^n \Delta r_i) = 0 \bmod(t^{n+1}).$$

We have a presentation of $I = (f_1, \dots, f_m)$:

$$0 \rightarrow R \xrightarrow{\beta} \bigoplus P(-n_i) \xrightarrow{\alpha} I \rightarrow 0 \quad (n_i = \deg f_i),$$

where α and β are homomorphisms of degree 0. Then $r_i(\underline{x}) \mapsto t^n$ -coefficient of $\sum f_i(\underline{x}; t)r_i(\underline{x}; t)$ defines an obstruction map $R \rightarrow P/I$. V extends to a V' if and only if the obstruction map lifts to a homomorphism $\delta: \bigoplus P(-n_i) \rightarrow P/I$. Both the obstruction map and its lifting δ (if it exists) are homomorphisms of degree 0. The obstruction map vanishes on R_0 , the submodule of R of trivial relations (i.e., those of the form $\alpha(a)b - \alpha(b)a$).

Define $T^2(I)$ by

$$\text{Hom}(\bigoplus P(-n_i), P/I)_0 \rightarrow \text{Hom}(R/R_0, P/I)_0 \rightarrow T^2(I) \rightarrow 0.$$

Then $T^2(I)$ contains the obstructions to lifting deformations from $k[t]/(t^n)$ to $k[t]/(t^{n+1})$, for any n .

We now restrict our attention to certain ideals defining X . Fix an I maximal among all (homogeneous) ideals defining X in P^n . $V(I)$ for any such I is called the *minimal cone* of X and is characterized by $(\underline{x}) \notin \text{Ass}(P/I)$. Since I is homogeneous, $I = \bigoplus_{j=0}^{\infty} I_j$, I_j = forms of degree j in I . Define $I_{(n)} = \bigoplus_{j=n}^{\infty} I_j$. The $I_{(n)}$ are also ideals defining X , while their cones contain $V(I)$.

Define $T_n^1 = \text{Hom}_P(I_{(n)}, P/I)_0$. If $n \leq m$ the natural inclusion $I_{(m)} \rightarrow I_{(n)}$ induces a map $T_n^1 \rightarrow T_m^1$.

PROPOSITION 2. If $(\underline{x}) \notin \text{Ass } P/I$, then $(\underline{x}) \notin \text{Ass}(\text{Hom}(M, P/I))$ for any graded P -module M .

PROOF. Suppose $(x) \in \text{Ass}(\text{Hom}(M, P/I))$ for some P -module M . Then there is a $\phi \in \text{Hom}(M, P/I)$ such that $(x) = \text{ann}(\phi)$. In particular $\phi \neq 0$, so that $\phi(m) \neq 0$ for some $m \in M$. Then $(x) \subseteq \text{ann}(\phi(m))$ and, being maximal, (x) must equal $\text{ann}(\phi(m))$, so that $(x) \in \text{Ass}(P/I)$.

COROLLARY 2.1. For $n \leq m$, $T_n^1 \rightarrow T_m^1$ is injective.

PROOF. Suppose not. Then there is a $\phi \in T_n^1$ that goes to the zero map in T_m^1 . Thus $\phi(b) = 0$ if $\deg b \geq m$. Therefore $(x)^m \phi = 0$ in T_n^1 so that $(x) \in \text{Ass}(\text{Hom}(I_{(n)}, P/I))$ which contradicts the proposition.

COROLLARY 2.2. If $(x) \notin \text{Ass}(P/I)$, the usual map

$$\alpha: \text{Hom}(M, P/I)_0 \rightarrow H^0((\text{Hom}(M, P/I))^\sim)$$

is injective (where $\tilde{M}$ means the sheaf associated to M).

PROOF. In [6, 67] Serre shows that the map $M \rightarrow \Sigma H^0(\tilde{M}(n))$ is injective if $(x) \notin \text{Ass}(M)$.

Thus, in particular, the following map is injective:

$$T_n^1 \rightarrow H^0((\text{Hom}(I, P/I))^\sim) = H^0(N_X) \quad \text{for any } n.$$

We will use these results to compare the deformation theory of the cone to that of X in $\mathbb{P}^n$.

Let S_n be the representative of the functor of deformations of the cone $V(I_{(n)})$. $\text{Spec } S_n$ is the parameter space for deformations of this cone.

Let S be the local ring of the Hilbert scheme $\mathbf{H}$ at the point corresponding to X . $\text{Spec } S$ parametrizes deformations of X over schemes $\text{Spec } T$, $T \in C$.

Let t (resp. t_n) denote the Zariski tangent space and O (resp. O_n) the obstruction space (see, e.g., [5, p. 153]) of $\text{Spec } S$ (resp. $\text{Spec } S_n$).

$$\begin{aligned} t &= H^0(N_X), & t_n &= T_n^1, \\ O &\hookrightarrow H^1(N_X), & O_n &\hookrightarrow T_n^2. \end{aligned}$$

Any deformation of the cone $V(I_{(n)})$ over $\text{Spec } T$, $T \in C$, induces a deformation of X over $\text{Spec } T$. This defines a map $\phi: \text{Spec } S_n \rightarrow \text{Spec } S$.

THEOREM 1. Let X be a local complete intersection. It is well known that there is an N such that for $n \geq N$, $(P/I)_n \xrightarrow{\sim} H^0(\mathcal{O}_X(n))$. For $n \geq N$, $\phi: \text{Spec } S_n \rightarrow \text{Spec } S$ is an isomorphism.

PROOF. For any n we have a presentation

$$0 \rightarrow R \rightarrow \bigoplus P(-n_i) \rightarrow I_{(n)} \rightarrow 0, \quad n_i \geq n.$$

Apply $\text{Hom}(\cdot, P/I)$ to get

$$0 \rightarrow \text{Hom}(I_{(n)}, P/I) \rightarrow \bigoplus P/I(n_i) \rightarrow \text{Hom}(R, P/I) \text{ exact.}$$

Take associated sheaves; then

$$0 \rightarrow N_X \rightarrow \bigoplus \mathcal{O}_X(n_i) \rightarrow \text{Hom}(R, P/I)^\sim \text{ is exact.}$$

Call the cokernel Q . Thus

$$0 \rightarrow N_X \rightarrow \bigoplus \mathcal{O}_X(n_i) \rightarrow Q \rightarrow 0 \quad (1)$$

is an exact sequence of $\mathcal{O}_X$ -modules.

Consider the exact sequence

$$0 \rightarrow \text{Hom}(I_{(n)}, P/I) \rightarrow \bigoplus P/I(n_i) \rightarrow \text{Hom}(R/R_0, P/I) \rightarrow S \rightarrow 0 \quad (2)$$

whose degree zero part is

$$0 \rightarrow T_n^1 \rightarrow \left(\bigoplus P/I(n_i) \right)_0 \rightarrow \text{Hom}(R/R_0, P/I)_0 \rightarrow T_n^2 \rightarrow 0.$$

Passing to associated sheaves

$$0 \rightarrow N_X \rightarrow \bigoplus \mathcal{O}_X(n_i) \rightarrow (\text{Hom}(R/R_0, P/I))^\sim \rightarrow \tilde{S} \rightarrow 0 \text{ is exact.}$$

Since X is a local complete intersection, and locally $\tilde{S}$ is T_2 , which is 0 [5, p. 150], it follows that $\tilde{S}$ is the zero sheaf. Therefore

$$0 \rightarrow N_X \rightarrow \bigoplus \mathcal{O}_X(n_i) \rightarrow (\text{Hom}(R/R_0, P/I))^\sim \rightarrow 0 \text{ is exact.} \quad (3)$$

From (1) and (3) we have $Q = (\text{Hom}(R/R_0, P/I))^\sim$.

Taking cohomology in (1) for the top row and taking (2) as the bottom row we have

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(N_X) & \rightarrow & \bigoplus H^0(\mathcal{O}_X(n_i)) & \rightarrow & H^0(Q) & \rightarrow & H^1(N_X) \\ & & \phi_1 \uparrow & & \phi_2 \uparrow & & \phi_3 \uparrow & & \phi_4 \uparrow \\ 0 & \rightarrow & T_n^1 & \rightarrow & \bigoplus (P/I)_{n_i} & \rightarrow & \text{Hom}(R/R_0, P/I)_0 & \rightarrow & T_n^2 & \rightarrow & 0 \end{array} \quad (4)$$

The maps ϕ_1 , ϕ_2 , and ϕ_3 are injective by Corollary 2.2. For $n \geq N$, ϕ_2 is an isomorphism. It follows easily that ϕ_1 is surjective (hence an isomorphism) and that ϕ_4 is injective.

Thus we have $t_n \xrightarrow{\sim} t$, and

$$\begin{array}{ccc} \mathcal{O}_n & \xrightarrow{\tilde{\phi}} & 0 \\ \downarrow & & \downarrow \\ T_n^2 & \rightarrow & H^1(N_X) \end{array}$$

so that $\tilde{\phi}$ is injective. By the proposition on p. 153 of [5] ϕ is an isomorphism. (Note that although the proposition is stated for local complex spaces the proof works over any field.)

This theorem means that the study of the local structure of the Hilbert scheme $\mathbf{H}$ is equivalent to the study of deformations of the cone $V(I_{(n)})$ for large enough n .

3. The unobstructed deformations of Mumford's curves. For one of Mumford's curves it is not hard to see that $N = 6$ (use Riemann-Roch). If γ is defined by $I(\eta, \xi)$ on $F = xA + yB$, then deformations of $V(I)$ represent the case $N = 3$. From diagram (4) we can compute that the cokernel of ϕ_1 is one-dimensional. Since $\dim H^0(N_\gamma) = 57$, the deformations of the cone $V(I)$ represent a 56-dimensional subfamily of the deformations of γ in $\mathbf{P}^3$.

The ideal I can be written in a determinantal form.

Let

$$M = \begin{bmatrix} x & y & B\xi & -A\xi \\ -B & A & x\eta & y\eta \end{bmatrix}.$$

Define M_{ij} = determinant of the 2×2 matrix formed by columns i and j . For convenience let $M = (a_{ij})$. The following two propositions are easily established.

PROPOSITION 3.1. (a) I is generated by the M_{ij} .

$$(b) \quad \begin{aligned} r_1 &= (-a_{13}, a_{12}, -a_{11}, 0), & r_2 &= (a_{14}, 0, a_{12}, -a_{11}), \\ r_3 &= (-a_{23}, a_{22}, -a_{21}, 0), & r_4 &= (-a_{24}, 0, a_{22}, -a_{21}). \end{aligned}$$

PROPOSITION 3.2. Let $N = (a_{ij})$ be a 2×4 matrix over any commutative ring. Then

$$\begin{aligned} -a_{13}N_{12} + a_{12}N_{13} - a_{11}N_{23} &= 0, & a_{14}N_{12} + a_{12}N_{23} - a_{11}N_{24} &= 0, \\ -a_{23}N_{12} + a_{22}N_{13} - a_{21}N_{23} &= 0, & -a_{24}N_{12} + a_{22}N_{23} - a_{21}N_{24} &= 0. \end{aligned}$$

Perturb A, B, x, y, ξ , and η in M as follows. Let

$$M(t) = \begin{pmatrix} x + tc & y + td & (B + tb)(\xi + te) & -(A + ta)(\xi + te) \\ -B - tb & A + ta & (x + tc)(\eta + tf) & (y + td)(\eta + tf) \end{pmatrix},$$

where a, b, c, d, e, f are homogeneous of the same degree as A, B, x, y, ξ, η , respectively. Let $I(t)$ be the ideal generated by the $M(t)_{ij}$. $I(0) = I$, and Proposition 2 gives the lifting of the relations r_1, r_2, r_3, r_4 . This gives a flat family of the cone $V(I)$ over $k(t)$, and hence an unobstructed deformation of γ in $\mathbb{P}^3$.

Any cubic form is in the ideal (x, y, A, B) . This, together with Proposition 1.2, guarantees that any element of $|4H + 2E|$ or any cubic is $M(1)$ for some choice of $a, \dots, f$.

Thus the $M(t)$ account for all deformations of $V(I)$ and of γ in $\mathbb{P}^3$ over the parameter space $\text{Spec } k(t)$.

4. Obstructed deformations. The dimension of Hilb at γ is 56, while the dimension of its tangent space, $H^0(N_\gamma)$, is 57. $H^0(N_\gamma)$ represents infinitesimal deformations of γ . There must be infinitesimal deformations that are obstructed.

$N_{X,Y}$ is the normal sheaf of X in Y . As in [3] define $N_F = N_{\gamma,F}$, $N_P = N_{F,\mathbb{P}^3}|_\gamma \cong \mathcal{O}_\gamma(3h)$, $h = H \cdot \gamma$. Let $N_\gamma = N_{\gamma,\mathbb{P}^3}$. The sequence $0 \rightarrow N_F \rightarrow N_\gamma \rightarrow N_P \rightarrow 0$ is exact. Thus $0 \rightarrow N_F \rightarrow N_\gamma \rightarrow \mathcal{O}_\gamma(3h) \rightarrow 0$ is exact. Taking cohomology; we get the exact sequence

$$0 \rightarrow H^0(N_F) \rightarrow H^0(N_\gamma) \rightarrow H^0(\mathcal{O}_\gamma(3h)) \rightarrow H^1(N_F) \rightarrow H^1(N_\gamma) \rightarrow H^1(\mathcal{O}_\gamma(3h)) \rightarrow 0.$$

$N_F \cong \mathcal{O}_\gamma(\gamma \cdot \gamma) \cong \mathcal{O}_\gamma(K_\gamma + h)$, hence $H^1(N_F) = 0$. Thus $0 \rightarrow H^0(N_F) \rightarrow H^0(N_\gamma) \rightarrow H^0(\mathcal{O}_\gamma(3h)) \rightarrow 0$ and $0 \rightarrow H^1(N_\gamma) \rightarrow H^1(\mathcal{O}_\gamma(3h)) \rightarrow 0$ are exact.

The sections of N_γ coming from sections of N_F correspond to infinitesimal deformations of γ in F , i.e., members of $|4H + 2E|$. These are unobstructed. Thus the obstructed deformations of γ must correspond to sections of N_γ coming from $H^0(\mathcal{O}_\gamma(3h))$. Those sections of $\mathcal{O}_\gamma(3h)$ that are given by one cubic polynomial on γ

(i.e., $\text{im}(H^0(\mathcal{O}_{\mathbf{P}^3}(3)) \rightarrow H^0(\mathcal{O}_\gamma(3h)))$ are also unobstructed. Thus an obstructed deformation must correspond to an element of $H^0(\mathcal{O}_\gamma(3h))$ not induced by an element of $H^0(\mathcal{O}_{\mathbf{P}^3}(3))$.

Since γ is nonsingular, the ring $\sum_{n=0}^\infty H^0(\mathcal{O}_\gamma(nh))$ is the integral closure of the projective co-ordinate ring of γ , P/I . Elements of $H^0(\mathcal{O}_\gamma(nh))$ may be thought of as elements of the quotient field K of P/I , of degree n , and integral over P/I .

Let $u = -B\xi/x$. ($\xi \neq 0$ or else γ is singular.)

$$\text{Deg } u = 3 \text{ in } K, \quad u^2 = B^2\xi^2/x^2 = -x^2\eta\xi/x^2 = -\eta\xi$$

so u is integral over P/I , but not in P/I . Thus u should lead to an obstructed deformation of γ . Note that

$$u = -B\xi/x = A\xi/y = -y\eta/A = x\eta/B \quad \text{in } K.$$

LEMMA 4.1. $(P/I)_n \rightarrow H^0(\mathcal{O}_\gamma(nh))$ is an isomorphism for $n \geq 6$, not for $n = 3, 4, 5$.

PROOF. Using the Riemann-Roch theorem we have

$$\dim H^0(\mathcal{O}_\gamma(3h)) = 20, \quad \dim H^0(\mathcal{O}_\gamma(nh)) = 14n - 23, \quad n > 3,$$

$$\dim(P/I)_n = \dim P_n - \dim I_n = \binom{n+3}{3},$$

and the ideal $I = (F, S_1, S_2, S_3)$, F cubic, S_i sextic. Thus

$$\dim I_n = \begin{cases} \binom{n}{3}, & n = 3, 4, 5, \\ 14n - 23, & n \geq 6. \end{cases}$$

Thus the map $(P/I)_n \rightarrow H^0(\mathcal{O}_\gamma(nh))$, injective since $(\underline{x}) \notin \text{Ass}(P/I)$, is surjective if $n \geq 6$, but not for $n = 3, 4, 5$.

PROPOSITION 4.2. The deformation theories of γ in $\mathbf{P}^3$ and of $V(I_{(n)})$ are the same for $n \geq 6$, different for $n < 6$.

PROOF. The lemma and Theorem 1 of §2 give the results for $n \geq 6$. For $n < 3$, $I_{(n)} = (F, S_1, S_2, S_3)$. Take the resolution $0 \rightarrow R \rightarrow P(-3) \oplus P(-6)^3 \rightarrow I_{(n)} \rightarrow 0$. The map

$$\phi_2: P_3 \oplus P_6^3 \rightarrow H^0(\mathcal{O}_\gamma(3h)) \oplus H^0(\mathcal{O}_\gamma(6h))^3$$

is not surjective by the lemma. Thus the map $\phi_1: T_n^1 \rightarrow H^0(N_\gamma)$ is not surjective. Thus the deformation theories are different.

The cases $n = 4, 5$ are similar.

Let $\underline{\sigma} = (x^3, x^2y, \dots, w^3)$ be the 20-tuple of cubic monomials in P . Then $I_{(6)} = (F\underline{\sigma}; S_1, S_2, S_3)$. $I_{(6)}$ has the resolution $0 \rightarrow R \rightarrow P(-6)^{23} \rightarrow I_{(6)} \rightarrow 0$.

R is generated by two types of relations:

- (i) relations among the cubics, i.e. $r = (r; 0, 0, 0)$, $r \cdot \underline{\sigma} = 0$;
- (ii) relations induced by those on I .

Let $B\xi = \underline{a} \cdot \underline{\sigma}$, $A\xi = \underline{b} \cdot \underline{\sigma}$, $\eta = \underline{c} \cdot \underline{\sigma}$; then the latter are generated by

$$\begin{aligned} r_1 &= (-\underline{a}; y, -x, 0), & r_2 &= (\underline{b}; 0, y, -x), \\ r_3 &= (-x\underline{c}; A, B, 0), & r_4 &= (-y\underline{c}; 0, A, B). \end{aligned}$$

A deformation of $V(I)$ is given by $\Delta f = (G, T_1, T_2, T_3)$, G cubic, T_i sextic. This yields a deformation of $I_{(6)}$ defined by $(G\underline{\sigma}; T_1, T_2, T_3)$.

For any cubic H , $Hu \in P$ (since $(x, y, A, B)u \in P$ and $H \in (x, y, A, B)$). By abuse of notation we consider $Hu \in P/I$. Thus $u\underline{\sigma} \in (P/I)^{20}$. We call $u\underline{\sigma}$ any choice of coset representatives of $u\underline{\sigma}$ in P^{20} . any two choices for $u\underline{\sigma}$ differ by an element of I^{20} .

Write $\eta = \alpha x + \beta y + \gamma A + \delta B$, $\deg \alpha = \deg \beta = 3$, $\deg \gamma = \deg \delta = 2$.

PROPOSITION 4.3. *Let $T_1 = (2\beta x - 2\gamma B)\xi$, $T_2 = (-\alpha x + \beta y + \gamma A - \delta B)$, $T_3 = (2\delta A - 2\alpha y)\xi$. For any choice of $u\underline{\sigma}$, $\Delta f = (u\underline{\sigma}; T_1, T_2, T_3)$ defines an infinitesimal deformation of $V(I_{(6)})$, hence of γ .*

PROOF. It is only necessary to check that $\Delta f \cdot r \in I$ for all $r \in R$. E.g.,

$$\begin{aligned} \Delta f \cdot r_4 &= -yu\underline{\sigma} \cdot \underline{c} + AT_2 + BT_3 = -yu\underline{\eta} + AT_2 + BT_3 \\ &= -y(-\alpha B\xi + BA\xi - \gamma y\eta + \delta xy) + AT_2 + BT_3 \pmod{I}, \end{aligned}$$

since $xu = B\xi \pmod{I}$, etc. It is then easy to see $\Delta f \cdot r_4 = 0 \pmod{I}$. The rest are similar.

Note that T_1, T_2, T_3 depend only on η and ξ , not on the choice of $u\underline{\sigma}$.

Any section of $H^0(\mathcal{O}_\gamma(3h))$ is given by $G + mu$, $G \in P_3$, $m \in k$. Thus any infinitesimal deformation of γ comes from a $\Delta f = ((G + mu)\underline{\sigma}, T'_1, T'_2, T'_3)$ where the T'_i are T_i plus the corresponding sextic for G (see §3).

5. The order of obstruction. The obstructions to extending infinitesimal deformations of γ lie in $H^1(N_\gamma) \xrightarrow{\sim} H^1(\mathcal{O}_\gamma(3h))$. It is convenient to observe that $H^1(\mathcal{O}_\gamma(3h))$ can be computed via a Koszul complex.

Define $D(f) = \mathbf{P}^3 \sim V(f)$, $f \in P$. Then $D(x), D(y), D(A), D(B)$ is an affine cover of γ , so that the cohomology of γ can be computed via the standard Čech resolution for this cover.

$L = \Sigma H^0(\mathcal{O}_\gamma(n))$ is the integral closure of P/I . Let $M = L(3)$; thus the associated sheaf $\tilde{M} = \mathcal{O}_\gamma(3h)$.

The sequence (x, y, A, B) is regular on P . Let $K = K.(x, y, A, B)$ be the graded Koszul complex

$$\begin{aligned} 0 \rightarrow P(-6) \rightarrow P(-5)^2 \oplus P(-4)^2 \rightarrow P(-2) \oplus P(-3)^4 \oplus P(-4) \\ \rightarrow P(-1)^2 \oplus P(-2)^2 \rightarrow P \end{aligned}$$

with the usual maps (see, e.g., [2, p. 245]). Let $C_\star$ be the complex $\text{Hom}(K, M)$, and $(C_\star)_0$ the subcomplex of $C_\star$ of degree zero.

PROPOSITION 5.1. $H^i((C_\star)_0) \xrightarrow{\sim} H^i(\mathcal{O}_\gamma(3h))$, $i = 0, 1$.

PROOF. We use the following notation:

$$\begin{array}{ccccccccccc} C_*: 0 & \rightarrow & M & \rightarrow & M(1)^2 \oplus M(2)^2 & \rightarrow & M(2) \oplus M(3)^4 \oplus M(4) & \rightarrow & M(4)^2 \oplus M(5)^2 & \rightarrow & M(6) & \rightarrow & 0 \\ & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \\ C_*: 0 & \rightarrow & M & \xrightarrow{d} & C_0 & \xrightarrow{d_0} & C_1 & \xrightarrow{d_1} & C_2 & \xrightarrow{d_2} & C_3 & \rightarrow & 0 \end{array}$$

thus $H^i((C_*)_0) = (\text{Ker } d_i)_0 / (\text{im } d_{i-1})_0$.

Taking associated sheaves we have the complex $\tilde{C}_*$:

$$0 \rightarrow \tilde{M} \xrightarrow{d'} \tilde{C}_0 \xrightarrow{d'_0} \tilde{C}_1 \xrightarrow{d'_1} \tilde{C}_2 \xrightarrow{d'_2} \tilde{C}_3 \rightarrow 0.$$

This complex is exact since $H^i(\tilde{C}_*) = \text{Ext}^i(\mathcal{O}_{\mathbb{P}^3}, \tilde{M})$, which is 0 for $i > 0$, and is $\tilde{M}$ for $i = 0$.

Let $Q = \text{im } d'_0 = \text{ker } d'_1$. Then $0 \rightarrow \tilde{M} \rightarrow \tilde{C}_0 \rightarrow Q \rightarrow 0$ is exact. Taking cohomology,

$$0 \rightarrow H^0(\tilde{M}) \rightarrow H^0(\tilde{C}_0) \rightarrow H^0(Q) \rightarrow H^1(\tilde{M}) \rightarrow 0.$$

The final zero is $H^1(\tilde{C}_0)$, zero since $H^1(\mathcal{O}_\gamma(nh)) = 0$ for $n \geq 4$ (nh is nonspecial).

The usual map $f: M \rightarrow \Sigma H^0(\tilde{M}(n))$ is an isomorphism. Thus in the following diagram (where $S = (\text{ker } d_1)_0 / (\text{im } d_0)_0$).

$$\begin{array}{ccccccccc} 0 & \rightarrow & H^0(\tilde{M}) & \rightarrow & H^0(\tilde{C}_0) & \rightarrow & H^0(Q) & \rightarrow & H^1(\tilde{M}) & \rightarrow & 0 \\ & & f \uparrow & & f_0 \uparrow & & f_1 \uparrow & & f_2 \uparrow & & \\ 0 & \rightarrow & (M)_0 & \rightarrow & (C_0)_0 & \xrightarrow{d_0} & (\text{ker } d_1)_0 & \rightarrow & S & \rightarrow & 0 \end{array}$$

f and f_0 are isomorphisms. We also have $0 \rightarrow Q \rightarrow \tilde{C}_1 \rightarrow \tilde{C}_2$ where

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(Q) & \rightarrow & H^0(\tilde{C}_1) & \rightarrow & H^0(\tilde{C}_2) \\ & & f_1 \uparrow & & f_3 \uparrow & & f_4 \uparrow \\ 0 & \rightarrow & (\text{ker } d_1)_0 & \rightarrow & (C_1)_0 & \rightarrow & (C_2)_0 \end{array}$$

where f_3 and f_4 are isomorphisms, hence f_1 is also an isomorphism. Then $S = H^1((C_*)_0) \xrightarrow{\sim} H^1(\mathcal{O}_\gamma(3h))$.

The curve γ is a local complete intersection, that is, there are $g_1, \dots, g_n \in P$ so that on $D(g_i) = W_i$, the curve γ is $V(f_i, s_i)$; with $f_i, s_i \in P_{(\mathbb{g})}$.

To find the element of $H^1(N_\gamma)$ that is the obstruction to lifting the deformation given by Δf , we change our point of view slightly. The curve γ is defined by the ideal (f_i, s_i) on $W_i = D(g_i)$. To extend the deformation of γ it is enough to find extensions of the deformation over each W_i by functions of degree zero such that the resulting perturbed ideals agree on the overlaps.

Let $W_{ij} = W_i \cap W_j$. We have (u_i, v_i) such that

$$(f_i + tu_i, s_i + tv_i) = (f_j + tu_j, s_i + tv_j) \pmod{t^2} \text{ over } W_{ij}.$$

We seek (u'_i, v'_i) such that

$$(f_i + tu_i + t^2u'_i, s_i + tv_i + t^2v'_i) = (f_j + tu_j + t^2u'_j, s_j + tv_j + t^2v'_j) \pmod{t^3}$$

on W_{ij} . Thus we want (u'_i, v'_i) such that there are a, b, c , etc. such that

$$\begin{aligned} f_j + tu_j + t^2u'_j &= (a + ta' + t^2a'')(f_i + tu_i + t^2u'_i) \\ &\quad + (b + tb' + t^2b'')(s_i + tv_i + t^2v'_i) \pmod{t^3} \\ &= af_i + bs_i + t(au_i + bv_i + a'f_i + b's_i) \\ &\quad + t^2(au'_i + bv'_i + a'u_i + b'v_i + a''f_i + b''s_i) \pmod{t^3} \end{aligned}$$

and

$$\begin{aligned} s_j + tv_j + t^2v'_j &= (c + tc' + t^2c'')(f_i + tu_i + t^2u'_i) \\ &\quad + (d + td' + t^2d'')(s_i + tv_i + t^2v'_i) \pmod{t^3} \end{aligned}$$

or

$$\begin{aligned} u'_j &= au'_i + bv'_i + a'u_i + b'v_i + a''f_i + b''s_i, \\ v'_j &= cu'_i + dv'_i + c'u_i + d'v_i + c''f_i + d''s_i. \end{aligned} \quad (1)$$

Note that

$$\begin{aligned} u_j &= au_i + bv_i + a'f_i + b's_i, \\ v_j &= cu_i + dv_i + c'f_i + d's_i. \end{aligned} \quad (2)$$

$$\begin{aligned} u'_j &= a'u_i + b'v_i + au'_i + bv'_i \\ v'_j &= c'u_i + d'v_i + cu'_i + dv'_i \pmod{(f_i, s_i)}. \end{aligned} \quad (3)$$

Now $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = N_{ji}$, so that

$$S = \left[\begin{pmatrix} a'u_i + b'v_i \\ c'u_i + d'v_i \end{pmatrix} \right] \in H^1(N_\gamma) \quad (4)$$

must vanish if we are to solve for

$$\begin{pmatrix} u'_i \\ v'_i \end{pmatrix} = \left(s_{ji} = \begin{pmatrix} u'_j \\ v'_j \end{pmatrix} - N_{ji} \begin{pmatrix} u'_i \\ v'_i \end{pmatrix} \right).$$

For Mumford's γ we know the map $H^1(N_\gamma) \rightarrow H^1(\mathcal{O}_\gamma(3h))$ given by

$$\left[\begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \right] \mapsto [u_{ij}],$$

where u_{ij} is the restriction of u_i to W_{ij} , is an isomorphism, so we only need to compute the $u_{ij} = a'u_i + b'v_i \pmod{(f_i, s_i)}$.

It is not hard to exhibit an open cover of $\mathbf{P}^3$, such that γ is defined by two equations on each open set.

PROPOSITION 5.2. *On $D(x)$, $(F/x^3, S_1/x^6)$ defines γ ; on $D(y)$, $(F/y^3, S_3/y^6)$; on $D(zA)$, $(F/zA, S_3/(zA)^2)$; on $D(wA)$, $(F/wA, S_3/(wA)^2)$; on $D(zB)$, $(F/zB, S_1/(zB)^2)$; on $D(wB)$, $(F/wB, S_1/(wB)^2)$.*

PROOF. The relation r_1 gives $-B\xi F + yS_1 - xS_2 = 0$. Thus, on $D(x)$,

$$S_2/x^6 = (B\xi/x^4)(F/x^3) + (y/x)(S_1/x^6)$$

so $S_2/x^6 \in (F/x^3, S_1/x^6)$. The rest follows similarly.

Let N_{21} be the transition matrix of the normal bundle on $D(x) \cap D(y)$. $N_{21} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the solutions of

$$F/y^3 = a(F/x^3) + b(S_1/x^6), \quad S_3/y^6 = c(F/x^3) + d(S_1/x^6) \pmod{I}.$$

Thus

$$N_{21} = \begin{bmatrix} x^3/y^3 & 0 \\ 2x^2A\xi/y^6 & x^4/y^4 \end{bmatrix}.$$

Consider $\Delta f = ((G + mu)\underline{\sigma}; T_1, T_2, T_3)$, $m \neq 0$, as in §4. Under the map $\phi_1: T_6^1 \rightarrow H^0(N_\gamma)$, f goes to

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix}, \quad i = 1, \dots, 6,$$

where $u_1 = -mB\xi/x^4 + G/x^3$, $u_2 = mA\xi/y^4 + G/y^3$. Note that the map $H^0(N_\gamma) \rightarrow H^0(\mathcal{O}_\gamma(3h))$ is given by projection on the first factor.

LEMMA 5.3. $u_{12} = mA\xi^2/xy^5 + G\xi/xy^4$, where u_{12} is the restriction of u_1 to W_{12} .

PROOF. From (2)

$$\begin{aligned} mA\xi/y^4 + G/y^3 &= (x^3/y^3)(-mB\xi/x^4 + G/x^3) \\ &\quad + a'(F/x^3) + b'(S_1/x^6). \end{aligned}$$

So

$$a'(F/x^3) + b'(S_1/x^6) = (x^2\xi/y^4)(F/x^3).$$

Take $a' = x^2\xi/y^4$, $b' = 0$. From (4)

$$u_{12} = a'u_1 + b'v_1 = mA\xi^2/xy^5 + G\xi/xy^4.$$

THEOREM 2. If $\xi \notin (x, y, A, B)$, then the cocycle $[u_{ij}] \neq 0$ in $H^1(\mathcal{O}_\gamma(3h))$, hence the infinitesimal deformation Δf does not extend to a deformation over $k[t]/(t^3)$.

PROOF. Under the isomorphism $f_2: H^1((C_0)_0) \rightarrow H^1(\mathcal{O}_\gamma(3h))$, $[u_{ij}]$ is the image of $[\phi_{ij}]$, where $\phi_{12} = mA\xi^2/y + G\xi = (u + G)\xi$. If $[\phi_{ij}] = 0$, then $\phi_{12} = (u + G)\xi = y\phi_1 - x\phi_2$, where $\phi_1, \phi_2 \in H^0(\mathcal{O}_\gamma(4h))$. For $C \in P$, $Cu \in P/I$ if and only if $C \in (x, y, A, B)$. Thus $y\phi_1 - x\phi_2 \in P/I$, while $u\xi \in P/I$. Thus $[\phi_{ij}] \neq 0$, hence $[u_{ij}] \neq 0$.

Thus if the curve γ is defined by the ideal $(Ax + By, x^2\eta + B^2\xi, xy\eta - AB\xi, y^2\eta + A^2\xi)$, with $\xi \notin (x, y, A, B)$, then the infinitesimal deformation given by Δf above does not extend to a deformation modulo t^2 , i.e., it is obstructed to the second order.

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