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Author(s): John Fogarty

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ALGEBRAIC FAMILIES ON AN ALGEBRAIC SURFACE.

By JOHN FOGARTY.*

0. Introduction. The purpose of this paper is to compute the cohomology of the structure sheaf of the Hilbert scheme of the projective plane. In fact, we achieve complete success only in characteristic zero, where it is shown that all higher cohomology groups vanish, and that the only global sections are constants.

The work falls naturally into two parts. In the first we are concerned with an arbitrary scheme, X, smooth and projective over a noetherian scheme, S. We show that each component of the Hilbert scheme parametrizing closed subschemes of relative codimension 1 on X, over S, splits in a natural way into a product of a scheme parametrizing *Cartier divisors,* and a scheme parametrizing subschemes of lower dimension. This fact is particularly useful in the case where X is an algebraic surface, in that it reduces the problem of families of subschemes to the problems of families of divisors, for which there is already an elaborate theory, (cf. [7]), and families of subschemes of dimension zero—which is the chief concern of the second part of the paper.

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1. Families of Cartier Divisoors. Let X be a projective scheme, flat over S. Fix a projective embedding of X over S, and define a presheaf of sets, $\mathbf{Div}_X$, on the category of schemes S' over S by:

$$\mathbf{Div}_X(S') = \left\{\begin{matrix}\text{set of Cartier divisors } D \\ \text{on } X \underset{S}{\times} S', \text{ flat over } S'\end{matrix}\right\}.$$

(A Cartier divisor D on $X \times S'$, flat over S' is called a relative Cartier divisor over S'. In the present paper all divisors are assumed to be *effective.*) Using the Hilbert polynomials of the closed subschemes D_s along the fibres over points, $s \in S'$, we can break up $\mathbf{Div}_X$ into disjoint subsheaves, $\mathbf{Div}^P{}_X$, where:

$$\mathbf{Div}^P{}_X(S') = \left\{\begin{matrix}\text{set of } D \in \mathbf{Div}_X(S') \text{ such that for} \\ \text{all } s \in S', \text{ and all } n \geqq 0, \\ \chi(\mathcal{O}_{D_s}(n)) = P(n)\end{matrix}\right\}.$$

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One also defines a presheaf:

$$\mathbf{Hilb}^P{}_X(S') = \begin{Bmatrix} \text{set of closed subschemes } Z \text{ of } X \times S' \\ \text{flat over } S', \text{ with } \chi(\mathcal{O}_{Z_s}(n)) = P(n), \\ \text{for all } s \in S', \text{ and all } n \geqq 0. \end{Bmatrix}$$

(cf. [2]). Then $\mathbf{Div}^P{}_X$ is represented by an open subscheme of the scheme representing $\mathbf{Hilb}^R{}_X$, which we denote by $\mathrm{Div}^P{}_X$ and $\mathrm{Hilb}^P{}_X$ respectively.

There is a canonical way to associate a Cartier divisor to any coherent cheaf $\mathcal{F}$ on a noetherian scheme X satisfying the following conditions: (i) $\mathcal{F}$ is a torsion sheaf, i.e., $\mathcal{F}_x = (0)$ for all $x \in X$ of depth 0. (ii) For each $x \in X$, $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite homological dimension. We denote this Cartier divisor by $\mathrm{Det}(\mathcal{F})$. For the construction and properties of $\mathrm{Det}(\mathcal{F})$, see [1] and [6]. The only properties that we will need are given by the following lemma whose proof can be found in [1].

LEMMA 1.1. *Let X be a noetherian scheme, $\mathfrak{I}$ an ideal in $\mathcal{O}_X$ such that $\mathcal{O}_X/\mathfrak{I}$ satisfies* (i) *and* (ii) *above. Then* $\mathrm{Det}(\mathcal{O}_X/\mathfrak{I})$ *is defined by the principal component, $\mathfrak{J}$, of $\mathfrak{I}$, i.e., locally, $\mathfrak{J}$ is the intersection of the primary components of $\mathfrak{I}$ belonging to points of depth* 1 *on X.*

PROPOSITION 1.2. *If X is smooth over S then* Div_X *is a closed subscheme of* Hilb_X.

Proof. Let $Z \subset X \underset{S}{\times} \mathrm{Hilb}_X$ be the "universal" closed subscheme. Since Z is flat and $X \times \mathrm{Hilb}_X$ is smooth over Hilb_X, $\mathrm{Det}(\mathcal{O}_Z)$ exists. Denote this divisor by D_Z. It is easily seen that D_Z is a relative divisor over Hilb_X, (cf. [1]). To see that Div_X is closed, choose $h \in \mathrm{Div}_X$, and let h' be any point of Hilb_X in the closure of h. Then $Z_h = (D_Z)_h$, and since both Z and D_Z are flat over Hilb_X, $Z_{h'}$ and $(D\ )_{h'}$ have the same Hilbert polynomial. By Lemma 1.1, the latter is a closed subscheme of the former. Hence they are equal, and $h' \in \mathrm{Div}_X$.

Now let S be a connected noetherian scheme, and let X be a smooth projective S-scheme. Fix a polynomial, P, with $\deg P = \dim X/S - 1$, and let H be a connected component of $\mathrm{Hilb}^P{}_{X/S}$. Let Y be a connected noetherian S-scheme, and let Z be a Y-valued point of H, i.e., Z is a closed subscheme of $X \times Y$, flat over Y, such that the corresponding morphism, $Y \to \mathrm{Hilb}_{X/S}$ factors through H.

Let $\mathfrak{J}$ be the invertible ideal in $\mathcal{O}_{X \times Y}$ defining $\mathrm{Det}(\mathcal{O}_Z)$. If $\mathfrak{I}$ is the ideal defining Z, let $\mathcal{K} = (\mathcal{I} : \mathcal{J})$. Then $\mathcal{I} \subset \mathcal{J}$, and $\mathcal{J}\mathcal{K} \subset \mathcal{I}$, so that—$x$ being a point of X—if $\mathcal{J}_x$ is generated by $f \in \mathcal{O}_{Y\times Y,x}$, then each $g \in \mathcal{I}_x$ is of the form fh for some $h \in \mathcal{O}_{Y\times Y,x}$. But then $h \in \mathcal{K}$. Therefore $\mathcal{I} \subset \mathcal{J}\mathcal{K}$, so that $\mathcal{I} = \mathcal{J}\mathcal{K}$.

Let Z' be the closed subscheme of $X \times Y$ defined by $\mathcal{K}$. Then one has an exact sequence on $X \times Y$:

$$\begin{array}{l} 0 \to \mathcal{J}/\mathcal{J}\mathcal{K} \to \mathcal{O}_Z \to \mathcal{O}_{D_Z} \to 0. \\ \qquad\ \wr\wr \\ \quad\ \mathcal{J} \otimes \mathcal{O}_{Z'} \end{array}$$

Since Z and D_Z are flat over Y, so is $\mathcal{I}/\mathcal{J}\mathcal{K}$, and since $\mathcal{J}$ is invertible, it follows that Z' is flat over Y. Since Y is connected, the Hilbert polynomial, P', of Z' along the fibres over Y is constant.

LEMMA 1.3. *Formation of the residual quotient, $\mathcal{K} = (\mathcal{I} : \mathcal{J})$ commutes with base extension, i.e., if $Y' \to Y$ is a morphism, and $X' = X \underset{Y}{\times} Y'$, $\mathcal{I}'$ defines $Z \times Y'$, and $\mathcal{J}'$ defines $D_Z \times Y'$, then $(\mathcal{I} : \mathcal{J})' = (\mathcal{I}' : \mathcal{J}')$, where $(\mathcal{I} : \mathcal{J})'$ is the inverse image of $(\mathcal{I} : \mathcal{J})$ on X'.*

Proof. One always has $(\mathcal{I} : \mathcal{J})' \subset (\mathcal{I}' : \mathcal{J}')$. But $\mathcal{J}'(\mathcal{I} : \mathcal{J})' = \mathcal{I}'$, and $\mathcal{J}'(\mathcal{I}' : \mathcal{J}') \subset \mathcal{I}'$, so $\mathcal{J}'(\mathcal{I}' : \mathcal{J}') = \mathcal{I}'$. The conclusion follows since $\mathcal{J}'$ is invertible.

Sending Z to the pair (D_Z, Z') as defined above yields a morphism of presheaves:

$$\Phi : h_H \to \mathbf{Div}^{P''}{}_{X/S} \times \mathbf{Hilb}^{P'}{}_{X/S},$$

where h_H stands, as usual, for the point functor $\mathrm{Hom}(\ ,H)$, of H, and where P'' is a polynomial with $\deg P = \deg P'' > \deg P'$. This defines a morphism of schemes, $\phi : H \to \mathrm{Div}^{P''} \times \mathrm{Hilb}^{P'}$.

Let $\mathfrak{D}$ (resp., $\mathfrak{H}$) be the connected component of $\mathrm{Div}^{P''}$ (resp., $\mathrm{Hilb}^{P'}$) in which the projection of the image of H lies. Let Z_1 be a Y-valued point of $\mathfrak{D}$, and Z_2 be a Y-valued point of $\mathfrak{H}$, defined by ideals, $\mathcal{J}_1$ and $\mathcal{I}_2$ in $\mathcal{O}_{X\times Y}$, respectively. Since $\mathcal{J}_1$ is invertible, $\mathcal{J}_1 \otimes (\mathcal{O}_{X\times Y}/\mathcal{I}_2)$ has constant Hilbert polynomial, P^*, along the fibres over Y. The exact sequence:

$$\begin{array}{l} 0 \to \mathcal{J}_1 \otimes \mathcal{I}_2 \to \mathcal{J}_1 \to \mathcal{J}_1 \otimes (\mathcal{O}_{X\times Y}/\mathcal{I}_2) \to 0 \\ \qquad\ \wr\wr \\ \qquad \mathcal{J}_1\mathcal{I}_2 \end{array}$$

shows that $\mathcal{O}_{X\times Y}/\mathcal{J}_1\mathcal{I}_2$ has constant Hilbert polynomial, P_0, along the fibres over Y.

Set $\mathcal{I} = \mathcal{J}_1\mathcal{I}_2$, and let Z_0 be the closed subscheme defined by this ideal. I claim that $P_0 = P$. Since the Hilbert polynomial of $p_{12}{}^*\mathcal{J}_1 \otimes p_{13}{}^*(\mathcal{O}_{X\times\mathcal{H}}/\mathcal{I}_2)$ is constant along the fibres of $X \times \mathcal{D} \times \mathcal{H}$ over $\mathcal{D} \times \mathcal{H}$, it follows from the exact sequence:

$$0 \to \mathcal{J}_1 \otimes (\mathcal{O}_{X\times Y}/\mathcal{I}_2) \to \mathcal{O}_{X\times Y}/\mathcal{I} \to \mathcal{O}_{X\times Y}/\mathcal{J}_1 \to 0$$

that the Hilbert polynomial of the middle term is independent of the choice of points of $\mathcal{D}$ and $\mathcal{H}$ corresponding to $\mathcal{J}_1$ and $\mathcal{I}_2$, respectively. Therefore since $P_0 = P$ for *some* choice of points, $P_0 = P$ for any choice of points. Therefore we have defined a morphism of presheaves:

$$\Psi\colon h_{\mathcal{D}} \times h_{\mathcal{H}} \to h_H.$$

To show that $\Phi \circ \Psi =$ identity on $h_{\mathcal{D}} \times h_{\mathcal{H}}$, note that in the second procedure, the fact that $\deg P' < \deg P < \dim X/S$ implies that $\mathcal{O}_{X\times Y}/\mathcal{I}_2$ has no associated points of depth $\leqq 1$ along the fibres over Y, so that $\mathcal{J}_1$ defines $\mathrm{Det}(\mathcal{O}_{X\times Y}/\mathcal{I}_2)$. Then $(\mathcal{I} : \mathcal{J}_1) = \mathcal{I}_2$, as required. For the converse, merely note that if $\mathcal{J}$ defines $\mathrm{Det}(\mathcal{O}_{X\times Y}/\mathcal{I})$, and $\mathcal{K} = (\mathcal{I} : \mathcal{J})$, then $\mathcal{J}\mathcal{K} = \mathcal{I}$, as before. Thus $\Psi \circ \Phi =$ identity on h_H. In summary:

THEOREM 1.4. *Let X be a smooth projective scheme over the connected noetherian scheme S. Let P be a numerical polynomial of degree equal to* $\dim X/S - 1$. *If $\mathcal{H}_1, \cdots, \mathcal{H}_q$ are the connected components of the scheme* $\mathrm{Hilb}^P{}_{X/S}$*—with respect to a fixed projective embedding of X over S—then there exist numerical polynomials, P'_i, P''_i, $1 \leqq i \leqq q$, with* $\deg P'_i < \deg P''_i$, *such that $\mathcal{H}_i$ is canonically isomorphic to $\mathcal{D}_i \times \mathcal{H}'_i$, where $\mathcal{D}_i$* (resp., $\mathcal{H}'_i$) *is a connected component of the scheme* $\mathrm{Div}^{P''_i}{}_{X/S}$ (resp., $\mathrm{Hilb}^{P'_i}{}_{X/S}$).

Remark. In general it is quite difficult to explicitly determine the polynomials P'_i and P''_i in terms of P. However in at least two important special cases it is quite easy. Assume for simplicity that S is the spectrum of a field, k.

1) If F is a nonsingular surface, then the Hilbert polynomials, P'_i are constants. If Z is a zero-dimensional closed subscheme of F, then for any invertible F-module, $\mathcal{L}$, we have $\mathcal{L} \otimes \mathcal{O}_Z \approx \mathcal{O}_Z$. It follows that $\mathrm{Hilb}^{dz+e}{}_{F/k}$ breaks up into a union of connected components of the form: $\mathcal{D} \times \mathrm{Hilb}^{e'}{}_{F/k}$, where $\mathcal{D}$ is a component of $\mathrm{Div}^{dz+e-e'}{}_{F/k}$. We prove in the next section that $\mathrm{Hilb}^{e'}{}_{F/k}$ is connected.

2) If X is a grassmannian (including the case of a projective space), then the degree of a divisor on X completely determines its Hilbert polynomial. In fact, it is easily seen that if $\deg P = \dim X - 1$, then

$$\mathrm{Hilb}^{P}{}_{X/k} \approx \mathrm{Div}^{P''}{}_{X/k} \times \mathrm{Hilb}^{P'}{}_{X/k},$$

where $P(z) = P''(z) + P'(z - \deg P)$.

2. Hilbert schemes of algebraic surfaces. Let X be a connected closed subscheme of $\boldsymbol{P}^N$ over the field k and let G be a unipotent algebraic group over k. Let σ be a k-homomorphism of G into $\boldsymbol{PGL}(N)$ such that X is stable under the action of G induced by σ.

PROPOSITION 2.1. *The set of fixed points of G on X for the action is connected.*

Proof. Clearly we may assume that k is algebraically closed. Since G has a composition series whose factors are copies of $\boldsymbol{G}_a$ we reduce to the case $G = \boldsymbol{G}_a$ by induction on the number of copies.

If $\dim X = 0$, there is nothing to prove. Assuming the result when X is a curve, let Z be the fixed point locus in question and assume that Z is not connected. Let C be a connected curve on X meeting two disjoint components of Z (C exists because, e.g., X is projective), and let Q be the Hilbert polynomial of C. σ induces an action of $\boldsymbol{G}_a$ on $\mathrm{Hilb}^{Q}{}_{X/k}$, since we may regard G_a as a subquotient of $\boldsymbol{PGL}(N)$. Let z be the closed point of Hilb^{Q} corresponding to C, and let U be the orbit of z for this induced action. The isotropy group for z is either finite or all of $\boldsymbol{G}_a$, so that either $U \approx \boldsymbol{G}_a$ or $U = \{z\}$. In the former case, $\bar{U}$ contains a unique closed point z_0 not in U. Then z_0 is a fixed point for the action of $\boldsymbol{G}_a$ on the Hilbert scheme, so that the corresponding curve C_0 is invariant under the action σ.

If x_0 and x_1 are two closed points of X lying on C, and in disjoint components of Z, then for all closed points $z' \in U$, the corresponding curve, C', is connected, and contains x_0 and x_1. Hence C_0 is connected, and contains the two given points.

Finally, to prove the result for a curve C, we use induction on the number of irreducible components. If C is irreducible, then either the action is trivial, or there is exactly one fixed point and C is a rational curve. If C is reducible, let C_0 be an irreducible component and let C' be the union of the irreducible components $\neq C_0$. Because of the orbit structure, each component is invariant, and $C_0 \cap C'$ consists of fixed points. By induction we may assume that each connected component of C' has a connected fixed point set. C_0 must then meet all of these fixed point sets. If C_0 does not consist entirely of fixed points, then as it contains only one fixed point, C' must be connected and the fixed point of C_0 lies in the fixed point set of C'. This proves the proposition.

PROPOSITION 2.2. *Let A be a finite dimensional local k-algebra. If $X = \operatorname{Spec} A$, then $\mathrm{Hilb}^n{}_{X/k}$ is connected for all n.*

Proof. Let $d = \dim_k A$. Then $\mathrm{Hilb}^n{}_{X/k}$ is a closed subscheme of the grassmannian, $\boldsymbol{G}$, parametrizing n-dimensional quotient spaces of the k-vector space A. If $\mathfrak{m}$ is the maximal ideal of A, the group $1+\mathfrak{m}$—under multiplication—is unipotent, and acts on $\boldsymbol{G}$ as follows.

The action of $1+\mathfrak{m}$ on A by multiplication gives a representation $\rho\colon 1+\mathfrak{m} \to \boldsymbol{SL}(d)$ from which we obtain a representation,

$$\overset{n}{\wedge}\rho\colon 1+\mathfrak{m} \to \boldsymbol{SL}(\binom{d}{n}),$$

the latter acting on $\overset{n}{\wedge} A$. $\boldsymbol{G}$ is a closed subscheme of $\boldsymbol{P}(\overset{n}{\wedge} A)$ and the canonical isogeny, $\boldsymbol{SL}(\binom{d}{n}) \to \boldsymbol{PGL}(\binom{d}{n} - 1)$ defines an action of $1+\mathfrak{m}$ on $\boldsymbol{P}(\overset{n}{\wedge} A)$ for which $\boldsymbol{G}$ is invariant. The fixed point of $\boldsymbol{G}$ for this action are those quotients, A/V such that $(1+\mathfrak{m})V = V$, i.e., V is an ideal of A. The support of $\mathrm{Hilb}^n{}_{X/k}$ being precisely this set of fixed points, it follows from 2.1 that $\mathrm{Hilb}^n{}_{X/k}$ is connected.

We recall now the construction of the "Hilbert-Chow" morphism in the special case of zero-dimensional subschemes. (For the general case, see [1], [6].) Let X be a closed subscheme of $\boldsymbol{P}^N$ over the field, k. We define a morphism, $\phi\colon (\mathrm{Hilb}^n{}_{X/k})_{red} \to \mathrm{Symm}^n(X)$, where $\mathrm{Symm}^n(X)$ denotes the n-th symmetric power of X (which exists because X is propjective), and which can be thought of as assigning to a closed subscheme Z of X, the 0-cycle consisting of the points of Z with multiplicities given by the lengths of their local rings on Z. In the category of schemes, $(\ )_{red}$ stands for the *reduced* object obtained from the object in brackets, (cf. [3]).

To define ϕ functorially, let T be a k-scheme, and let Z be a T-valued point of $\mathrm{Hilb}^n{}_{X/k}$ (Z is a closed subscheme of $X \times T$, etc.). Consider the diagram

$$\begin{array}{ccc} & \boldsymbol{H}\times T & \\ \swarrow f & & g \searrow \\ Z \subset \boldsymbol{P}^N \times T & & \hat{\boldsymbol{P}}^N \times T, \end{array}$$

where $\hat{\boldsymbol{P}}^N$ is the dual projective space,, and $\boldsymbol{H}$ is the incidence correspondence. Set $D_Z = \mathrm{Det}(g_* f^* \mathcal{O}_Z)$. One checks easily that D_Z is a relative Cartier divisor over T, and that for each $t \in T$, $(D_Z)_t$ is exactly the Chow form of the 0-cycle corresponding to Z_t. The formation of D_Z in this way commutes with base extension, and we obtain a morphism ψ of $\mathrm{Hilb}^n{}_{X/k}$ into a projective space of divisors on $\boldsymbol{P}^N$ whose image is supported on the Chow variety of 0-cycles of degree n on X, i.e., $\mathrm{Symm}^n(X)$. Then we let $\phi = \psi_{red}$.

Proposition 2.3. *If X is connected, then $\mathrm{Hilb}^n{}_{X/k}$ is connected.*

Proof. It suffices to show that the closed fibres of the morphism ϕ defined above are connected. If $\sum_{i=1}^{t} n_i z_i$ is a geometric point of $\mathrm{Symm}^n(X)$ with the z_i distinct points of X, and $\sum n_i = n$, then—set theoretically—the fibre of $\mathrm{Hilb}^n{}_{X/k}$ over this point is:

$$\prod_{i=1}^{t} \mathrm{Hilb}^{n_i}{}_{Z_i/k}$$

where $Z_i = \mathrm{Spec}\, \mathcal{O}_{X,z_i}/\mathfrak{m}^{n_i}{}_{X,z_i}$. By Proposition 2.2, each factor is connected, and so the product is connected.

Theorem 2.4. *Let F be a nonsingular surface over the field k. Then $\mathrm{Hilb}^n{}_{F/k}$ is a nonsingular scheme of dimension $2n$.*

Proof. By 2.3, $\mathrm{Hilb}^n{}_{F/k}$ is connected. But the open subset of $\mathrm{Hilb}^n{}_{F/k}$ whose geometric points correspond to reduced subschemes of F evidently has dimension $2n$. Therefore $\mathrm{Hilb}^n{}_{F/k}$ has at least one irreducible component of dimension $2n$. The theorem will then be proved if we show that the Zariski tangent space to $\mathrm{Hilb}^n{}_{F/k}$ has dimension $\leqq 2n$ at *any* point, for then every point on the known component must be simple on $\mathrm{Hilb}^n{}_{F/k}$, and there can be no other components.

If z_0 is a point of $\mathrm{Hilb}^n{}_{X/k}$, and Z_0 is the corresponding closed subscheme of F, defined by the ideal $\mathcal{J} \subset \mathcal{O}_F$, let $\mathrm{Supp}(Z_0) = \{x_1, \cdots, x_r\}$. Then

$$\mathrm{Hom}_{\mathcal{O}_F}(\mathcal{J}, \mathcal{O}_F/\mathcal{J}) \approx \prod_i \mathrm{Hom}_{\mathcal{O}_{F,x_i}}(\mathcal{J}_{x_i}, \mathcal{O}_{Z_0,x_i})$$

and the module on the left is precisely the Zariski tangent space to $\mathrm{Hilb}^n{}_{F/k}$ at z_0. Therefore we are reduced to proving the following lemma.

Lemma 2.5. *Let A be a 2-dimensional regular local ring and let I be an ideal of A primary for the maximal ideal, $\mathfrak{m}$. If length $A/I = n$, then length $\mathrm{Hom}_A(I, A/I) \leqq 2n$.*

Proof. Since A is regular, and I is a torsion-free A-module, I has homological dimension 1 over A. To see this note that if k is the residue field of A, then $\mathrm{Tor}_2{}^A(I, k) = (0 : \mathfrak{m})_I = (0)$. Since I is not principal, it is not free, so the homological dimension cannot be 0 or 2. Since I is an ideal, it has rank 1, so there is an exact sequence:

$$0 \to A^{(r)} \to A^{(r+1)} \to I \to 0,$$

$(r+1 = \dim I/\mathfrak{m}I)$. From this we obtain the exact sequence:

$$0 \to \mathrm{Hom}_A(I, A/I) \to \mathrm{Hom}_A(A^{(r+1)}, A/I) \to \mathrm{Hom}_A(A^{(r)}, A/I) \to \mathrm{Ext}_A{}^1(I, A/I) \to 0.$$

Since length $\mathrm{Hom}(A^{(r)}, A/I) = r \cdot$ length A/I, it suffices to show, using the fact that the Euler characteristic of the above sequence is zero, that $\mathrm{Ext}_A{}^1(I, A/I)$ has length $\leqq n$. Now from the exact sequence: $0 \to I \to A \to A/I \to 0$, we obtain an isomorphism: $\mathrm{Ext}_A{}^1(I, A/I) \approx \mathrm{Ext}_A{}^2(A/I, A/I)$. Hence to prove length $\mathrm{Ext}_A{}^2(A/I, A/I) \leqq n$ will suffice.

Since A is a 2-dimensional regular local ring, $\mathrm{Ext}_A{}^2(A/I, \ )$ is a right exact functor. Therefore there is a surjection:

$$\mathrm{Ext}_A{}^2(A/I, A) \to \mathrm{Ext}^2(A/I, A/I),$$

so that we are finally reduced to showing that $\mathrm{Ext}_A{}^2(A/I, A)$ has length $\leqq n$. But, in fact, we have equality here, for, by local duality (cf. [4]), $\mathrm{Ext}_A{}^2(\ , A)$ is an exact functor on the category of A-modules of finite length, so we have $\mathrm{Ext}_A{}^2(k, A) \approx k$, where k is the residue field, and we obtain our result by induction on the length.

COROLLARY 2.6. *The morphism,* $\phi\colon \mathrm{Hilb}^n{}_{F/k} \to \mathrm{Symm}^n(F)$ *is birational.*

Proof. This follows immediately from the fact that ϕ induces an isomorphism of the open set in Hilb^n corresponding to reduced subschemes with the open set in Symm^n corresponding to 0-cycles without multiple components, together with the fact that Hilb^n is nonsingular.

COROLLARY 2.7. *If F is a nonsingular surface with $H^1(F, \mathcal{O}_F) = (0)$, then* $\mathrm{Hilb}_{F/k}$ *is a disjoint union of nonsigular schemes each of which has the form* $\boldsymbol{P} \times \mathrm{Hilb}^n{}_{F/k}$, *where* $\boldsymbol{P}$ *is a projective space.*

Proof. It is known that for any variety V with $H^1(V, \mathcal{O}_V) = (0)$ Div_V is a disjoint union of projective spaces, (cf. [7]). The corollary then follows from Theorem 1.4.

By a *generalized curve* we mean a scheme, X, which has a cover by affine open subschemes, U_i, each of which is isomorphic to the spectrum of a Dedekind domain.

LEMMA 2.8. *Let Y be a generalized curve, and let $f: X \to Y$ be a morphism, such that for all $y \in L$, $f^{-1}(y)$ is a geometrically integral scheme of dimension d over $\kappa(y)$. Then f is flat.*

Proof. For this we may evidently assume that Y is the spectrum of a discrete valuation ring. Flatness is then equivalent to the assertion that the generic fibre *specializes* to the special fibre, i.e., each component of X dominates Y. But the generic fibre specializes in any case to a closed subscheme of dimension d of the special fibre. Since the fibres of f are integral the result follows. Combining Theorem 2.4 with the lemma, we get:

THEOREM 2.9. *Let X be a smooth scheme of relative dimension 2 over a generalized curve, Y. Then $\mathrm{Hilb}^n{}_{X/Y}$ is smooth of relative dimension $2n$ over Y.*

COROLLARY 2.10. $\mathrm{Hilb}_{\boldsymbol{P}^2/\boldsymbol{Z}}$ *is smooth over* $\boldsymbol{Z}$.

Mattuck has proved, [10], that $\mathrm{Symm}^n(\boldsymbol{P}^N)$ is a rational variety. Therefore, by 2.6, $\mathrm{Hilb}^n{}_{\boldsymbol{P}^2/k}$ is a nonsingular rational variety. It follows that there exists a nonsingular rational variety, U, (cf. [5]), and birational morphisms:

$$\begin{tikzcd} & U \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ \mathrm{Hilb}^n{}_{\boldsymbol{P}^2/k} & & \boldsymbol{P}^{2n} \end{tikzcd}$$

provided that k has characteristic 0. It is known, [5], that $R^i g_{j*}\mathcal{O}_U = (0)$, $i > 0$, $j = 1, 2$. From the Leray spectral sequence for g_2 it follows that $H^i(U, \mathcal{O}_U) = (0)$, for $i > 0$. Then by the Leray sequence for g_1, we get: $H^i(\mathrm{Hilb}^n{}_{\boldsymbol{P}^2/k}, \mathcal{O}_{\mathrm{Hilb}^n{}_{\boldsymbol{P}^2/k}}) = (0)$, for $i > 0$.

THEOREM 2.11. *The structure sheaf of* $\mathrm{Hilb}_{\boldsymbol{P}^2/k}$ *has no higher cohomology if k has characteristic zero. In any characteristic, its global sections are constants.*

Remark. Combining 2.10 and 2.11 it follows that $\mathrm{Hilb}_{\boldsymbol{P}^2}$ has no higher cohomology for almost all values of the characteristic.

3. Further results and comments. If F is a nonsingular surface over the field k, then one can show that F and $\mathrm{Hilb}^n{}_{F/k}$ have the same Picard *variety* (= reduced component of the identity in the Picard scheme) as follows. From the definition of the Albanese variety it follows that F and $\mathrm{Symm}^n(F)$ have the same Albanese variety, (cf. [9]). On the other hand, the Albanese is a birational invariant, so that $\mathrm{Symm}^n(F)$ and $\mathrm{Hilb}^n{}_{F/k}$ have the same Albanese variety. Since both F and $\mathrm{Hilb}^n{}_{F/k}$ are normal, their

Picard varieties are the duals of their Albanese varieties (cf. [8]). From this we obtain the following facts:

i) If k has characteristic 0, then $H^1(F, \mathcal{O}_F)$ and $H^1(\mathrm{Hilb}^n{}_{F/k}, \mathcal{O}_{\mathrm{Hilb}^n{}_{F/k}})$ are isomorphic.

ii) If k has characteristic 0 and $H^1(F, \mathcal{O}_F) = (0)$, then if H is a connected component of $\mathrm{Hilb}^P{}_{F/k}$, for any polynomial P, then $H^1(H, \mathcal{O}_H) = (0)$, and if k is algebraically closed, $H^0(H, \mathcal{O}_H) = k$.

The first assertion follows from a result of Cartier, (cf. [7]), which says that a group scheme of finite type over a field of characteristic 0 is reduced. The second follows from what we have already proved.

One is led to conjecture that ii) above holds also in characteristic $p > 0$, but Theorem 2.11 is the best I can do in this direction at present. One should also remark that these results do not go over *in toto* to higher dimensions. The analogue of our Theorem 2.4 is false in dimension 3, e.g., $\mathrm{Hilb}^4{}_{\boldsymbol{P}^3/\boldsymbol{C}}$ has singular points—if x, y, z, t are homogeneous cordinates, the Zariski tangent space at the point corresponding to the ideal $(x^2, y^2, z^2, xy, xz, yz)$ has dimension 18, but the generic dimension is evidently 12. I conjecture that $\mathrm{Hilb}^n{}_{\boldsymbol{P}^N/k}$ is always a variety—reduced and irreducible—but all we know is that it is connected.

For $n > 2$, the universal closed subscheme, $Z_n \subset \boldsymbol{P}^2 \times \mathrm{Hilb}^n{}_{\boldsymbol{P}^2/k}$ has singular points, for Z_n is a locally trivial fibre space over $\boldsymbol{P}^2$, with fibre of dimension $2n - 2$. However, e.g., for $n = 3$, the Zariski tangent space to the fibre at a point of type (x^2, xy, y^2) is 6—x, y, z being homogeneous coordinates in $\boldsymbol{P}^2$. Theorem 2.4 must therefore be regarded as a truly isolated phenomenon.

Finally we remark that everything that we have proved for $\boldsymbol{P}^2$ over $\boldsymbol{Z}$ we can prove for such "easy" surfaces as $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ over $\boldsymbol{Z}$. We leave to the reader the task of formlating the appropriate theorems.

HARVARD UNIVERSITY AND
UNIVERSITY OF PENNSYLVANIA.

REFERENCES.

[1] J. Fogarty, *Truncated Hilbert Functors* (to appear).

[2] A. Grothendieck, *Seminaire Bourbaki*, Expose 221, Paris, 1960.

[3] A. Grothendieck and J. Dieudonne, *Elements de Géomètrie Algebrique*, Publications Mathematiques Institut des Hautes Etudes Scientifiques, Paris, 1960.

[4] R. Hartshorne, *Residues and Duality*, Mathematical Notes, no. 20, Springer, Berlin, 1966.

[5] H. Hironaka, "Resolution of singularities of algebraic varieties over groundfields of characteristic zero," *Annals of Mathematics*, vol. 79, no. 2, March 1964.

[6] D. Mumford, *Geometric Invariant Theory*, Ergebnisse der Mathematik N.F. Bd. 34, Springer, Berlin, 1965.

[7] ———, *Lectures on Curves on an Alegbraic Surface*, Annals of Mathematics Studies, Study 59, Princeton, 1966.

[8] C. Chevalley, "Sur la Theorie de la variete de Picard," *American Journal of Mathematics*, vol. 82 1960), p. 435.

[9] S. Lang, *Abelian Varieties*, Interscience, New York, 1958.

[10] A. Mattuck, "The field of multi-symmetric functions," to appear in *Proceedings of the American Mathematical Society*.