



Further Pathologies in Algebraic Geometry

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FURTHER PATHOLOGIES IN ALGEBRAIC GEOMETRY.*¹

By DAVID MUMFORD.

The following note is not strictly a continuation of our previous note [1]. However, we wish to present two more examples of algebro-geometric phenomena which seem to us rather startling. The first relates to characteristic p behaviour, and the second relates to the hypothesis of the completeness of the characteristic linear system of a maximal algebraic family. We will use the same notations as in [1].

I.

The first example is an illustration of a general principle that might be said to be indicated by many of the pathologies of characteristic p :

A non-singular characteristic p variety is analogous to a general non-Kähler complex manifold; in particular, a projective embedding of such a variety is not as “strong” as a Kähler metric on a complex manifold; and the Hodge-Lefschetz-Dolbeault theorems on sheaf cohomology break down in every possible way.

In this case we wish to look at the two dimensional cohomology of an algebraic surface F , non-singular, and of any characteristic but 0. The surface we shall choose will (a) be specialization of a characteristic 0 surface F' , and (b) will satisfy $q = h^{0,1} = h^{1,0}$. Consequently the second Betti number B_2 is the same, whether defined (i) as that of F' in the topological sense, (ii) as $h^{2,0} + h^{1,1} + h^{0,2}$, or (iii) following Igusa [2], as $\text{Deg}(c_2) + 4q - 2$. Let ρ be the base number of F . Igusa showed that, in fact, $B_2 \geq \rho$. However, in characteristic 0, one has the stronger result, $B_2 = h^{2,0} + h^{1,1} + h^{0,2} \geq 2p_g + \rho$ (where $p_g = h^{2,0} = h^{0,2}$ is the geometric genus of F) as a result of the Hodge-Dolbeault theorems. Therefore the question arises whether this stronger inequality is valid in characteristic p . The answer is no.

A rather complicated example was discovered in 1961 by J. Tate and A. Ogg. Here is a very simple example: let E be a super-singular elliptic

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curve of characteristic p (i.e. such that the rank of $\text{End}(E)$ is 4). Let $F = E \times E$. In this case, in fact:

$$\rho = B_2 = 6; \quad p_g = 1.$$

Here $p_g = 1$ since the sheaf $\Omega_F^2 \cong \mathcal{O}_F$; and $B_2 = 6$ by Igusa's definition, for example, since $\text{Deg}(c_2) = 0$, and $q = 2$. Finally $\rho = 6$ since in general, for any two elliptic curves E_1 and E_2 , one knows that the base number ρ for $E_1 \times E_2$ equals 2 plus the rank of $\text{Hom}(E_1, E_2)$.

There remains one outstanding conjecture still neither proven nor disproven in characteristic p , which according to the general principle mentioned above ought to be false. This is the Regularity of the Adjoint, which may be stated as follows: if V is a non-singular projective surface and if H is a non-singular hyperplane section, then

$$H^1(\mathcal{O}_V) \rightarrow H^1(\mathcal{O}_H)$$

is injective.

II.

The second example concerns space curves in characteristic 0. Let A be any family of non-singular space curves, and let $a \in A$ represent the curve $\gamma \subset P_3$. Let T_a denote the Zariski tangent space to A at a , and let N denote the sheaf of sections of the normal bundle to γ in P_3 . Then it is well-known [3] that there is a "characteristic" map:

$$T_a \rightarrow H^0(N).$$

The problem of completeness consists in asking when, for given γ , there is a family A containing γ such that the characteristic map is surjective. Kodaira [3] has shown that such a family exists if $H^1(N) = (0)$. Our example shows that if $H^1(N) \neq (0)$, then there need not be such a family.

In fact, in our example, this incompleteness holds for every curve in an open set of the corresponding Chow variety. Consequently, it is also an example where the Hilbert scheme [4] has a multiple component, i.e. is not reduced at one of its generic points. Another corollary of this example is obtained by blowing up such a space curve $\gamma \subset P_3$ to a surface E in a new three-dimensional variety V_3 . Then Kodaira [5] has shown essentially that the local moduli scheme of the variety V_3 is isomorphic to the germ of the

Hilbert scheme of P_3 at the point corresponding to γ . Therefore we have constructed a non-singular projective three-dimensional variety whose local moduli scheme is nowhere reduced; in other words, any small deformation of V_3 is a variety the number of whose moduli is less than the dimension of $H^1(\Theta)$ (where Θ is the sheaf of vector fields).

The curves γ that we have in mind have degree 14 and genus 24. In the following, h will stand for the divisor class on γ induced by plane sections, and K_γ will stand for the canonical divisor on γ ; also F will stand for a cubic or quartic surface in P_3 , and H will stand for the (Cartier) divisor class on F induced by plane sections. The first step is partial classification of all space curves of this degree and genus, which confirms the results of M. Noether's well-known table [6].

(A) *Any non-singular space curve γ of degree 14 and genus 24 is contained in a pencil P of quartic surfaces.*

Proof. Since $\text{Deg}(4h) = 56$, and $\text{Deg}(K_\gamma) = 46$, the linear system $|4h|_\gamma$ is non-special,² and has dimension $56 - 24 = 32$. Since there is a 34-dimensional family of quartics in P_3 , (A) follows.

There are 2 cases: (a) the pencil has no fixed components, and (b) the pencil has fixed components. In case (a), note first that if F' and F'' span P , then $F' \cdot F'' = \gamma + c$, where c is a conic. Now c has at most double points, hence $\gamma + c$ has at most triple points. Therefore no point x is a double point for both F' and F'' . Noting that both F' and F'' are non-singular and transversal along $\gamma - c$, hence at all but a finite number of points of γ , it follows that almost every $F \in P$ is non-singular everywhere along γ .

(B) *Every algebraic family of space curves of type (a) has dimension less than or equal 56.*

Proof. It is enough to show that every family of pairs (γ, F) consisting of such curves γ , and quartics $F \supset \gamma$, F being non-singular along γ , has dimension at most 57. Now since all such quartics contains conics, they are not generic [7], and there is at most a $34 - 1 = 33$ dimensional family of quartics F involved in such a family of pairs. Moreover, the dimension of the set of all γ on one such F can be computed from the Riemann-Roch theorem on F :

² Here and below, $|D|_V$ always means the linear system on V in which the Cartier divisor D varies. Also, $(D^2)_V$ always denotes the self-intersection of D , as a divisor class on D (assuming D effective).

$$\dim |\gamma|_F = \frac{\text{Deg}(\gamma^2)_F}{2} + 1 + \{\dim H^1(\mathcal{O}_F(\gamma)) - \dim H^2(\mathcal{O}_F(\gamma))\}.$$

But $(\gamma^2)_F \equiv K_\gamma$ on γ , hence $\text{Deg}(\gamma^2)_F = 46$. Moreover, $H^i(\mathcal{O}_F(\gamma))$ is dual to $H^{2-i}(\mathcal{O}_F(-\gamma))$ by Serre duality. This cohomology group can be computed from the exact sequence:

$$0 \rightarrow \mathcal{O}_F(-\gamma) \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_\gamma \rightarrow 0.$$

It follows that both are zero, hence $\dim |\gamma|_F = 24$. Therefore, indeed, the set of pairs (γ, F) has dimension at most $33 + 24 = 57$.³

Now consider case (b). Such a γ must be contained in a reducible quartic, hence in a plane, a quadric, or a cubic surface. The first two possibilities are readily checked and it happens that they contain no curves of the required degree and genus. Moreover, such a curve is contained in a *unique* cubic surface F , because $\text{Deg}(\gamma) = 14 > 9 = \text{Deg}(F' \cdot F'')$, for two distinct cubic surfaces F' and F'' . We will say that γ is of type (b_0) if the cubic F is non-singular; otherwise, we will say that γ is of type (b_1) .

(C) *Every maximal algebraic family of curves γ of type (b_0) has dimension 56.*

Proof. Let γ be a curve of type (b_0) , and let F be the corresponding cubic surface. Since $K_F \equiv -H$, by the Riemann-Roch theorem on F :

$$\dim |\gamma|_F = \frac{\text{Deg}(\gamma \cdot \gamma + H)_F}{2} + \{\dim H^1(\mathcal{O}_F(\gamma)) - \dim H^2(\mathcal{O}_F(\gamma))\}.$$

But $K_\gamma \equiv \gamma \cdot (\gamma + K_F)$, hence $46 = \text{Deg}(\gamma^2)_F - \text{Deg}(\gamma \cdot H)_F = \text{Deg}(\gamma^2)_F - 14$, hence $\text{Deg}(\gamma^2)_F = 60$. Also, $H^i(\mathcal{O}_F(\gamma))$ is dual to $H^{2-i}(\mathcal{O}_F(-H-\gamma))$, and this group can be computed from the exact sequence:

$$0 \rightarrow \mathcal{O}_F(-H-\gamma) \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_{(H+\gamma)} \rightarrow 0.$$

Since $H + \gamma$ is a reduced and connected curve, $H^0(\mathcal{O}_{H+\gamma}) = k$ (constants), and this implies $H^i(\mathcal{O}_F(-H-\gamma)) = (0)$ for $i = 1$ and 2 . Putting all this together, we see that $\dim |\gamma|_F = 37$. Since there is a 19-dimensional family

³ It may be objected that we have used the Riemann-Roch theorem, and Serre duality as though F were non-singular. But since F is non-singular along γ , the former can be proved by means of the exact sequence:

$$0 \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_F(\gamma) \rightarrow \mathcal{O}_\gamma((\gamma^2)_F) \rightarrow 0.$$

And the latter can be proven either (a) directly by resolving the singularities of F and comparing the cohomology on F and on its resolution, or (b) as a consequence of Grothendieck's general theory [8]. In the second case, one merely has to note that F is always a Cohen-Macaulay variety; and since it is a *quartic* surface the canonical sheaf is simply \mathcal{O}_F itself.

of cubic surfaces, (C) follows if we show that a generic γ in a maximal algebraic family is contained in a generic cubic surface. But let $\gamma \subset F$ be any curve of the family. Then recalling that the divisor class group of any non-singular cubic surface is the same as that of any other, it follows that if the set of all non-singular cubic surfaces are suitably parametrized the invertible sheaf $\mathcal{O}_F(\gamma)$ will be a specialization of an invertible sheaf L defined on the generic cubic surface F^* . And since $H^i(\mathcal{O}_F(\gamma)) = (0)$ for $i=1$ and 2 , by the upper semi-continuity of cohomology [9], we conclude that $H^i(L) = (0)$ for $i=1$ and 2 , and that all sections of $\mathcal{O}_F(\gamma)$ are specializations of sections of L . Therefore $\dim H^0(L) = 38$; and since almost all sections of $\mathcal{O}_F(\gamma)$ are non-singular, so are almost sections of L . Hence there is a non-singular $\gamma^* \subset F^*$ specializing to $\gamma \subset F$. QED.

Now suppose C is the Chow variety of non-singular curves of degree 14, and genus 24. Let $C_b \subset C$ be the locus of curves of type (b) , and let $C_{b_1} \subset C_b$ be the locus of curves of type (b_1) . Then it is clear that C_b and C_{b_1} are closed (possibly reducible) subvarieties of C . By (B) and (C), every component of $C - C_b$ has dimension ≤ 56 , and every component of $C_b - C_{b_1}$ has dimension $= 56$. Therefore if C_0 equals C minus C_{b_1} and minus the closure of $C - C_b$, C_0 is an open set in the Chow variety, of dimension 56, and parametrizing *almost all* curves of type (b_0) .

We shall now single out a set of components of C_0 such that, if N is the normal sheaf to a γ in one of these components, then $\dim H^0(N) = 57$. In fact, we say that $\gamma \subset F$ is of type (b'_0) if there is a line E on F such that $\gamma \equiv 4H + 2E$ on F . Then the corresponding $C'_0 \subset C_0$ which is the locus of such curves is clearly closed in C_0 . But it is also open: if $\gamma^* \subset F^*$ specializes to $\gamma \subset F$, and if $\gamma \equiv 4H + 2E$ on F , then first of all, there is a line $E^* \subset F^*$ (possibly only rationally defined after a suitable base extension) which specializes to E ; and secondly, since the divisor class group is discrete and constant for all non-singular cubics,

$$\gamma - 4H - 2E \equiv 0 \text{ implies } \gamma^* - 4H^* - 2E^* \equiv 0.$$

Therefore γ^* is of type (b'_0) .

(D) If $\gamma \subset F$ is of type (b'_0) , then $\dim H^0(N) = 57$.

Proof. Let N_F be the sheaf of normal vector fields to γ and in F , and let N_P be the sheaf of normal vector fields to F , and in P_3 , which are defined along γ . Then we have the sequence:

$$0 \rightarrow N_F \rightarrow N \rightarrow N_P \rightarrow 0.$$

But if D is a non-singular divisor on a non-singular variety V , then its normal sheaf is isomorphic to $\mathcal{O}_D((D^2)_V)$. Therefore $N_F \cong \mathcal{O}_\gamma((\gamma^2)_F)$ and $N_F \cong \mathcal{O}_\gamma(3h)$. But since $K_\gamma \equiv (\gamma^2)_F + \gamma \cdot K_F \equiv (\gamma^2)_F - h$, it follows that $(\gamma^2)_F$ is a non-special divisor, of degree 60 in fact. Therefore $H^1(N_F) = (0)$ and $\dim H^0(N_F) = 60 - (24 - 1) = 37$. On the other hand, by the Riemann-Roch theorem for curves,

$$\begin{aligned} \dim H^0(\mathcal{O}_\gamma(3h)) &= 42 - (24 - 1) + \dim H^0(\mathcal{O}_\gamma(K_\gamma - 3h)) \\ &= 19 + \dim H^0(\mathcal{O}_\gamma((\gamma^2)_F - 4h)) \\ &= 19 + \dim H^0(\mathcal{O}_\gamma(2\gamma \cdot E)) \end{aligned}$$

(using the hypothesis $\gamma \equiv 4H + 2E$). But now, use the exact sequence:

$$0 \rightarrow \mathcal{O}_F(-4H) \rightarrow \mathcal{O}_F(2E) \rightarrow \mathcal{O}_\gamma(2\gamma \cdot E) \rightarrow 0.$$

It is readily seen that $H^i(\mathcal{O}_F(-4H)) = (0)$ for $i = 0$ and 1 , and that $\dim H^0(\mathcal{O}_F(2E)) = 1$. Putting all this information together we conclude: $\dim H^0(N) = 37 + 19 + 1 = 57$. QED.

It remains only to note:

- (E) *If F is any non-singular cubic surface, and $E \subset F$ is any line, there exist non-singular curves $\gamma \in |4H + 2E|$, and they have degree 14 and genus 24.*

Proof. The degree and genus of such a γ are computed by the usual formulae, recalling that $\text{Deg}(E^2)_F = -1$. To see that such a γ exists, it suffices, by the characteristic 0 Bertini theorem, to prove that $|4H + 2E|$ has no base points. But the only possible base points are the points of E , and we use the exact sequence:

$$0 \rightarrow \mathcal{O}_F(4H + E) \rightarrow \mathcal{O}_F(4H + 2E) \rightarrow \mathcal{O}_E(2) \rightarrow 0.$$

Since the sections of $\mathcal{O}_E(2)$ have no base points, it suffices to prove $H^1(\mathcal{O}_F(4H + E)) = (0)$. But this follows from the sequence:

$$0 \rightarrow \mathcal{O}_F(4H) \rightarrow \mathcal{O}_F(4H + E) \rightarrow \mathcal{O}_E(3) \rightarrow 0,$$

since $H^1(\mathcal{O}_F(4H)) = (0)$, and $H^1(\mathcal{O}_E(3)) = (0)$. QED.

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