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THE RADIUS OF THE HILBERT SCHEME

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Throughout, let k be an algebraically closed field of characteristic 0, and let $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$ denote the Hilbert scheme of subschemes of $\mathbf{P}^n_k$ with Hilbert polynomial $p(z)$. While much is known about specific Hilbert schemes, e.g. punctual Hilbert schemes (see Iarrobino [7] for a survey), or Hilbert schemes of curves with certain characteristics (see., e.g., Ciliberto and Sernesi [3], Piene and Schlessinger [10]), the general structure of such Hilbert schemes remains largely a mystery. While Hilbert schemes of hypersurfaces and of points in $\mathbf{P}^2$ are irreducible, most Hilbert schemes consist of more than one irreducible component. In fact, except for these cases, any Hilbert scheme having at least one point corresponding to a smooth subscheme must have at least two components [11]. The component structure, i.e. the number of irreducible components, their dimensions, how they intersect, and (where possible) what subschemes they parametrize, is not well understood.

The most important theorem to date along these lines is:

Theorem (Hartshorne [6]). $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$ *is connected.*

His proof can be broken into three major steps. In the first step, he connects any ideal to a Borel-fixed ideal (a monomial ideal fixed by the action of the group of upper triangular matrices on $k[x_0, \cdots, x_n]$) through a sequence of degenerations. From this Borel-fixed ideal, he then produces a new ideal whose degenerations include the original Borel-fixed ideal. This new ideal, termed a "fan", is produced via a two-step process: one first converts the Borel-fixed ideal to a square-free (polarized) ideal by adding a variable for each successive power of each original variable. One then pulls the result back to an ideal in the original variables by taking a linear section of the polarization. The resulting ideal defines a union of linear spaces, hence the term "fan". Finally, Hartshorne connects these fans to "tight" fans which, in turn, he connects to each other.

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Below, we improve this plan, using Galligo's Theorem [4] to make the degeneration to a Borel-fixed ideal in one step, and Gröbner basis arguments to vastly reduce the sequence of degenerations required in the third step. These improvements lead to the main theorem:

Theorem. *The radius of the incidence graph of irreducible components of the Hilbert scheme* $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$ *is at most* $d+1$, *where* d *is the dimension of the subschemes parametrized.*

Furthermore, we show that the lexicographic ideal (the extreme Borel-fixed ideal) is the true center of these degenerations. As a consequence, we easily recover Gotzmann [5] and Bayer's [1] result that the lexicographic ideal has the worst regularity of any ideal on the Hilbert scheme.

The importance of Borel-fixed ideals in the proofs of the theorems stated above stems from the fact that every component and every intersection of components contains at least one Borel-fixed ideal. This fact raises the following questions:

1. Is the subset of Borel-fixed ideals on a component enough to determine the component?
2. Is there a combinatorial method of determining when a subset of Borel-fixed ideals lies on a single component?

While the first question remains unanswered, we give a partial answer to the second question in Section 4 of the paper. In Section 2, we review the facts needed about Borel-fixed ideals. In Section 3, we recall the notion of fan introduced in Hartshorne's thesis and give an algorithm to compute the ideal of the fan associated to a Borel-fixed ideal. We also produce a simple, combinatorial method for determining the number of linear subspaces of each dimension in a fan associated to a Borel-fixed ideal. In Section 5, we prove the main theorem and several corollaries.

2. Borel-fixed ideals

For those unfamiliar with Borel-fixed ideals, this section contains the basic facts about Borel-fixed ideals used in the proofs of Theorems 6 and 7.

Convention. All Borel-fixed ideals will be *listed* in descending lexicographic order, by which we mean, $\{x^{A_1}, x^{A_2}, \cdots, x^{A_r}\}$, where $x^{A_1} \geq x^{A_2} \geq \cdots \geq x^{A_r}$ in pure (without reference to degree) lexicographic order.

Definition. For $f \in S$, let $\mathrm{in}_>(f)$ be the initial term of f with respect to a multiplicative order $>$. Let $\mathrm{in}_>(I) = \langle \mathrm{in}_>(f) | f \in I \rangle$.

Definition. The Borel group $B(n+1)$ is the group of all upper triangular $(n+1)\times(n+1)$ matrices. $(a_{ij})\in B(n+1)$ acts on $k[x_0,\cdots,x_n]$ via the map $x_i\mapsto\sum_{j=0}^n a_{ij}x_j$. A Borel-fixed ideal is an ideal which is fixed under the action of $B(n+1)$ on $k[x_0,\cdots,x_n]$.

The following properties of Borel-fixed ideals will be needed in the sequel:

1. All Borel-fixed ideals are monomial ideals. In particular, Bayer and Stillman [2] showed that in characteristic zero, a monomial ideal I is Borel-fixed if and only if whenever $x_1^{p_1}\cdots x_n^{p_n}\in I$, then for each $1\le j<i\le n$ and $0\le q\le p_i$, $x_1^{p_1}\cdots x_j^{(p_j+q)}\cdots x_i^{(p_i-q)}\cdots x_n^{p_n}\in I$.

2. If $I=\langle M_1,\cdots,M_s\rangle$ is a Borel-fixed ideal, then the saturation $\operatorname{sat}(I)$ of I is generated by $\{M_1|_{x_n=1},\cdots,M_s|_{x_n=1}\}$.

3. **Theorem 1** (Galligo [4]). *Let $>$ be a multiplicative order on S, $I\subset S$ a homogeneous ideal. There exists a Zariski-open subset $U\subset GL(n+1)$ such that for any $g\in U$, $\operatorname{in}_>(g\cdot I)$ is Borel-fixed.*

4. **Theorem 2** (see, e.g., Bayer [1]). *Given $I\subseteq S$ and a multiplicative order $>$ on S, there exists a flat family $J=\{I_t\}\subseteq S[t]$ of ideals such that*

(1) *$I_t\cong I$ by a scaling of variables $(I_1=I)$,*

(2) $\lim_{t\to 0} I_t=\operatorname{in}_>(I)$.

Remarks. 1. The second fact shows that subschemes determined by Borel-fixed ideals with Hilbert polynomial $p(z)$ are in one-to-one correspondence with those Borel-fixed ideals having Hilbert polynomial $p(z)$ in which x_n does not appear in any monomial in the minimal generating set.

2. Taken together, facts three and four imply that every component and intersection of components of the Hilbert scheme contains at least one Borel-fixed ideal.

These properties make Borel-fixed ideals a useful, combinatorial method of investigating the component structure of the Hilbert scheme. In the next section we explore one more nice property of Borel-fixed ideals.

3. Hartshorne's fans

This section recalls the notion of a "fan" as defined in Hartshorne's thesis [6], and the relationship of such fans to Borel-fixed ideals.

Notation. $\{t_{ij}\}$ and $\{z_{ij}\}$ are infinitely many independent indeterminates over $k(x_0,\cdots,x_n)$, $A=k[t_{ij}]_m$, $m=(t_{ij})$, and $P=A[z_{ij}]$.

Definition. Let I_g be the unique minimal monomial generating set of

I. The *polarization* $\overline{I} \subseteq P$ of $I \subseteq k[x_0, \cdots, x_n]$ is

$$\overline{I} = \left\langle \prod_{i=0}^{n} \prod_{j=1}^{s_0} z_{ij} | x_0^{s_0} \cdots x_{n-1}^{s_{n-1}} \in I_g \right\rangle .$$

Definition. Let $\sigma : P \to k[x_0, \cdots, x_n]$ be defined by $\sigma(z_{ij}) = l_{ij}$, where l_{ij} is a linear form in $k[x_0, \cdots, x_n]$, and let $I \subseteq k[x_0, \cdots, x_{n-1}]$ be a saturated monomial ideal. The fan associated to I under σ is the linear section, $\sigma(\overline{I})$, of $\overline{I}$.

Remarks. 1. All irreducible components of a fan are linear.

2. For sufficiently generic σ, the resulting fans are reduced. In particular, they have no embedded components.

3. For special σ and suitable I (e.g. I Borel-fixed), these fans correspond precisely to those defined in Hartshorne's thesis, as we will see below.

4. For sufficiently generic σ, there exists a flat deformation from $\sigma(\overline{I})$ to I.

If I is a saturated Borel-fixed ideal, it is possible to "read off" from I the geometry of the associated fan:

Definition. $\mathrm{sat}_{x_i}(I)$ is defined to be the ideal obtained by setting $x_i = 1$ in I. Inductively, $\mathrm{sat}_{x_{i_1}, x_{i_2}, \cdots, x_{i_r}}(I)$ is defined to be the ideal $\mathrm{sat}_{x_{i_1}}(\mathrm{sat}_{x_{i_2}, \cdots, x_{i_r}}(I))$.

Note. The definition of $\mathrm{sat}_{x_i}(I)$ given above corresponds to the usual definition of saturation if $i = n$ and I is a Borel-fixed ideal.

Theorem 3. *For I a Borel-fixed ideal, the number of components in the fan of I supported on the linear subspace defined by $(x_0, \cdots, x_s)$ for any s is the sum of the exponents of x_s occurring in the minimal generating set of* $\mathrm{sat}_{x_{s+1}, x_{s+2}, \cdots, x_n}(I)$.

To prove this, we need to delve into Hartshorne's original construction of a fan.

Proposition 1 (4.4 in Hartshorne [6]). *$\overline{I}$ is the irredundant intersection of those prime ideals $p = (z_{i_0 j_0}, \cdots, z_{i_s j_s})$ such that*

1. *$i_0, \cdots, i_s$ are all distinct.*
2. *$I \subseteq (x_{i_0}^{j_0}, \cdots, x_{i_s}^{j_s})$.*
3. *I is not contained in any ideal generated by a proper subset of the $x_{i_k}^{j_k}$.*

Note. The number and dimensions of the primary ideals in 2 satisfying all three conditions determine the number and dimensions of the linear subspaces of the fan associated to I, as we will see below.

Definition. The *canonical distraction* I' of I is defined to be the image of $\overline{I}$ under the map $\sigma : z_{ij} \mapsto x_i - t_{ij}x_n$.

Theorem 4 (4.9 in Hartshorne [6]). *The canonical distraction I' of I is an intersection of prime ideals of the form $p = (x_{i_0} - t_{i_0, j_0}x_n, \cdots, x_{i_s} - t_{i_s, j_s}x_n)$.*

Hartshorne originally defined a fan by a condition on the decomposition of its ideal into primes:

Definition (Hartshorne [6]). A *fan* X in $\mathbf{P}^n_k$ is a subscheme whose ideal I can be written as an intersection of prime ideals of the form $p = (x_0 - a_0x_n, \cdots, x_q - a_qx_n)$ for various $q \in \{0, \cdots, n-1\}$, and various $a_0, \cdots, a_q \in k$.

Using this definition and the above theorems we have:

Theorem 5 (4.10 in Hartshorne [6]). *For I a Borel-fixed ideal, the canonical distraction of I defines a fan $X'' \subseteq \mathbf{P}^n_K$ (K is the quotient field of $k[t_{ij}]$), which specializes linearly (see Hartshorne [6]) to the subscheme X defined by I ($X \subseteq \mathbf{P}^n_k$).*

Definition. The *fan decomposition* of a Borel-fixed ideal I is defined to be the set of primes p (described in Theorem 4) whose intersection is the canonical distraction of I.

An algorithm for producing the primary ideals mentioned in Proposition 1 is presented in Figure 1 (next page).

Notation. *Input*: $I = \{X^{A_1}, \cdots, X^{A_r}\} \subseteq k[x_0, \cdots, x_n]$ a minimal generating set for a Borel-fixed ideal, listed in descending lexicographic order.

Output: $\{p_l\}$ $l = 1, \cdots, N$ a set of ideals each consisting of monomials of the form $x_j^{b_j}$ for various j's. Throughout the sequel, p_l will denote an ideal produced by this algorithm.

Intermediate: p-current, partial p_l for some l. Monomials in p will be indicated by $x_j^{c_j}$ for various j's. $I' = \{x^{A_j} \in I | x^{A_j} \notin p\}$; I' is listed in descending lexicographic order. Denote the first monomial in I' by $x_0^{b_0} \cdots x_{n-1}^{b_{n-1}}$.

Initially: $p = (X^{A_1} = x_0^{a_0})$, $j = 0$, $l = 1$.

Given the primes, it is a simple matter to produce the fan itself by replacing x_i^j with $x_i - t_{i,j}x_n$ in each ideal, and then intersecting the resulting ideals. The above algorithm will be used in the proof of Theorem 3.

Next we prove that the algorithm is correct.

Proposition 2. *Given I a Borel-fixed ideal, let p be an ideal satisfying*

```
p = (x_0^{a_0})
while p ≠ ∅
    begin
        compute I';
        if (I' = ∅) then
            begin
                p_l = p; l = l + 1;
                while j ≥ 0 and c_j = 1
                    p = p\x_j^{c_j}; j = j − 1;
                if (j ≥ 0) then p = p\x_j^{c_j} ∪ {x_j^{c_j−1}};
            end
        else
            begin
                j = j + 1; p = p ∪ {x_j^{b_j}};
            end
    end
```

FIGURE 1. Algorithm to compute the prime ideals of Proposition 1 for a Borel-fixed ideal

1. $I \subseteq p \equiv (x_0^{c_0}, \cdots, x_s^{c_s})$
2. *I is not contained in any ideal generated by a proper subset of the $x_i^{c_i}$.*

Then p is produced by the algorithm. Furthermore, all p_l's produced by the algorithm satisfy these two properties.

Proof. That all p_l's produced by the algorithm satisfy property 1 is clear. To see that all p_l's satisfy property 2, note first that if $x_s^{c_s}$ is the last generator added to p_l, then $I \not\subseteq p_l \backslash x_s^{c_s}$, since otherwise $p_l \backslash x_s^{c_s}$ would have been produced by the algorithm. More generally, $x_i^{c_i} \in p_l$ implies $\{x_0^{c_0}, \cdots, x_{i-1}^{c_{i-1}}\}$ does not contain I. But suppose

$$\{x_0^{c_0}, \cdots, x_{i-1}^{c_{i-1}}, x_{i+1}^{c_{i+1}}, \cdots, x_s^{c_s}\}$$

contains I. Then for any $x^B \in I$ not in $\{x_0^{c_0}, \cdots, x_{i-1}^{c_i-1}\}$, there must exist some $x_j^{c_j} | x^B$, with $j > i$. But then $x_i^{c_i} x_j^{c_j} | x^B$, and hence, by the Borel property, $x_i^d \in I$, for some d. Thus

$$\{x_0^{c_0}, \cdots, x_{i-1}^{c_{i-1}}, x_{i+1}^{c_{i+1}}, \cdots, x_s^{c_s}\}$$

could not contain all of I.

For the first part of the proposition, we need the following lemma.

Lemma 1. *Define I', as in the algorithm, to be those minimal generators of I not divisible by $(x_0^{c_0}, \cdots, x_i^{c_i}) :=$ some initial subset of generators for a p_l. Then (assuming $I' \neq \emptyset$) the first monomial in I' in descending lexicographic order has x_{i+1} as the last variable occurring to a positive power.*

Proof. Let x^A be the first monomial in I' in descending lexicographic order. Let $N = \sum_{j=i+1}^{n-1} a_j$, and $x^{A'} = x_0^{a_0+t_0} \cdots x_i^{a_i+t_i} x_{i+1}^{t_{i+1}}$ where

1. $\sum_{j=0}^{i+1} t_j = N$
2. $a_j + t_j = \max\{b_j | x^B \in I'\}$ for $j = 0, \cdots, r$, for some $r \leq i$
3. if $r < i$, $a_{r+1} + N - \sum_{j=0}^{r} t_j < \max\{b_{r+1} | x^B \in I'\}$
4. if $r = i$, $t_{i+1} = N - \sum_{j=0}^{i} t_j$
5. $t_{r+2} = \cdots = t_{n-1} = 0$.

Now $x^{A'} \in I$ by the Borel property, and $x^{A'}$ is not divisible by $(x_0^{c_0}, \cdots, x_i^{c_i})$ (since none of the x^B's were). So there exists $x^C \in I'$ such that $x^C | x^{A'}$. Since $x^{A'} \geq x^A$ in descending lexicographic order, $x^C \geq x^A$ as well. But x^A is the first monomial in I' in descending lexicographic order, so $x^A = x^C$, i.e. $x^A | x^{A'}$. Since x^A and $x^{A'}$ are of the same degree, $x^A = x^{A'}$. Finally, $I \subseteq p$ implies $I' \subseteq (x_{i+1}^{c_{i+1}}, \cdots, x_s^{c_s})$, which implies $t_{i+1} = a_{i+1} > 0$. q.e.d.

Proof of Proposition 2 *continued.* By induction on the monomials in p_l. First note $c_0 \leq b_0^1$, where $x_0^{b_0^1}$ is the first monomial in I. Start with $p = (x_0^{c_0})$. Then I' has a first monomial $x_0^{b_0^2} x_1^{b_1^2}$ for some $b_0^2 < c_0$. Since $p \supseteq I$ and $x_0^{c_0}$ does not divide $x_0^{b_0^2} x_1^{b_1^2}$, it must be that $x_1^{c_1} | x_0^{b_0^2} x_1^{b_1^2}$, i.e. $c_1 \leq b_1^2$. Note that for a given c_0, and for each $c_1 \leq b_1^2$, the algorithm produces an initial $p = \{x_0^{c_0}, x_1^{c_1}\}$. Also all initial sequences of powers produced in the algorithm always appear in some p_l. So there exists a p_l produced by the algorithm having $(x_0^{c_0}, x_1^{c_1})$ as its first two monomials.

In general, suppose there exists a p_l produced by the algorithm matching the hypothesized p in the first $(x_0^{c_0}, \cdots, x_j^{c_j})$ monomials. Consider I'. By Lemma 1, the first monomial of I' is of the form $x_0^{b_0^t} \cdots x_j^{b_j^t} x_{j+1}^{b_{j+1}^t}$. $p \supseteq I$ implies $c_{j+1} \leq b_{j+1}^t$. But all such c_{j+1}'s are considered in the course of the algorithm. So there is a p_l having $(x_0^{c_0}, \cdots, x_{j+1}^{c_{j+1}})$ as its first monomials. q.e.d.

The proof of Theorem 3 now follows easily from the next lemma.

Lemma 2. *Let I be a Borel-fixed ideal whose generators only involve the variables $x_0, \cdots, x_r$. The sum of the exponents of x_r occurring in a minimal generating set of I equals the number of components of the fan supported on the linear subspace of $\mathbf{P}^n$ defined by $(x_0, \cdots, x_r)$.*

Proof. Note that in the algorithm, since only the variables $x_0, \cdots, x_r$ occur in I, once we have added in x_r to some power, the resulting p must contain the entire ideal. In the proof of 2, we saw that x_r to some power gets added into p precisely when the first monomial in I' has x_r to that power occurring as its last variable. Now, every monomial in the minimal generating set of I occurs as an initial monomial of I' at some time during the execution of the algorithm:

Given $x^A = x_0^{a_0} \cdots x_l^{a_l}$, $p = (x_0^{a_0+1}, \cdots, x_{l-2}^{a_{l-2}+1}, x_{l-1}^{a_{l-1}+1})$ divides all monomials preceding x^A but not x^A since, if $x^B > x^A$, where $x^B = x_0^{b_0} \cdots x_r^{b_r}$, then $b_i > a_i$ for some $i \in [0, \cdots, l-1]$. So this p gives I' with x^A initial.

In particular, every monomial in the minimal generating set of I having x_r to a positive power occurs as the first monomial of I' at some point. Finally $(x_0^{j_0}, \cdots, x_r^{j_r}) \supseteq I$ and no subset contains I, so $(x_0^{j_0}, \cdots, x_{r-1}^{j_{r-1}}, x_r^k) \supseteq I$ and no subset contains I for $0 \le k \le j_r$. Thus for each $x^A = x_0^{a_0} \cdots x_r^{a_r}$, we get precisely a_r components supported on the linear subspace of $\mathbf{P}^n$ defined by $(x_0, \cdots, x_r)$. q.e.d.

Proof of Theorem 3. Since $I = \bigcap p_l$, the intersection of all of the p_l's produced by the algorithm,

$$\mathrm{sat}_{x_{s+1}, x_{s+2}, \cdots, x_n}(I) = \mathrm{sat}_{x_{s+1}, x_{s+2}, \cdots, x_n}\left(\bigcap p_l\right).$$

But the right-hand side consists precisely of those p_l which correspond to subschemes supported on the linear subspace defined by $(x_0, \cdots, x_j)$, for $j \le s$. Thus, the p_l's involving precisely the variables $(x_0, \cdots, x_j)$ for $j \le s$ coincide for I and $\mathrm{sat}_{x_{s+1}, x_{s+2}, \cdots, x_n}(I)$. The corollary now follows from Lemma 2, since $\mathrm{sat}_{x_{s+1}, x_{s+2}, \cdots, x_n}(I) \subseteq k[x_0, \cdots, x_s]$. q.e.d.

4. The partition

This section presents a theorem which gives an easy method of partitioning Borel-fixed ideals into classes, each of which must lie in a single component.

Theorem 6. *Let $\mathbf{P}_k^n = \operatorname{Proj} S$, $S = k[x_0, \cdots, x_n]$, k an algebraically closed field of characteristic 0. Let $p(z)$ define a given Hilbert polynomial,* *and let* $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$ *be the associated Hilbert scheme. Then the set*

$$\mathscr{I} = \{I | I \subseteq S \text{ is Borel-fixed with Hilbert polynomial } p(z), \text{ and } \mathrm{sat}_{x_{n-1}, x_n}(I) = L\}$$

for some Borel-fixed ideal L, *consists of ideals defining subschemes in* $\mathbf{P}^n_k$ *which all lie on a single component of* $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$.

Before proving this theorem, we need the following lemma.

Lemma 3. *Let* I_1, I_2 *be the ideals of fans of two Borel-fixed ideals in* $k[x_0, \cdots, x_n]$ *having the same Hilbert polynomial, and suppose* $I_1 = J \cap N_1$, $I_2 = J \cap N_2$ *where* J *corresponds to all fan components whose dimension is* $\geq i$, *so that* N_j $(j = 1, 2)$ *corresponds to fan components whose dimension is* $\leq i - 1$. *Let* F_1 *and* F_2 *be the corresponding fans. Then the number of* $(i-1)$*-dimensional components of* F_1 = *the number of* $(i-1)$*-dimensional components of* F_2.

Proof. Assume the number of $(i-1)$-dimensional components is nonzero in at least one of the N_j's. (Otherwise the number of $(i-1)$-dimensional components is 0 for both fans, and the lemma holds trivially.) Consider the exact sequence

$$0 \to S/I_j \to S/J \oplus S/N_j \to S/(J + N_j) \to 0.$$

Since no component of the fan defined by N_j is embedded in any component of the fan defined by J, we know the intersection of the fans corresponding to J and N_j has dimension strictly less than $i - 1$. Thus the Hilbert polynomial $P_{S/(J+N_j)}$ has degree $< i - 1$.

Now we consider leading coefficients. The fan corresponding to N_j has dimension $(i-1)$ for at least one of $j = 1, 2$. So P_{S/N_j} has degree $(i-1)$. By the exact sequence above, we have $P_{S/N_j} = P_{S/I_j} - P_{S/J} + P_{S/(J+N_j)}$ for $j = 1, 2$. But P_{S/I_j} is the same for $j = 1, 2$, $P_{S/J}$ is the same for $j = 1, 2$, and $P_{S/(J+N_j)}$ has degree $< i - 1$, so the coefficient of the degree $(i-1)$-term in P_{S/N_j} is the same for $j = 1, 2$. Since there is no term of degree i or above in one (and hence both) of P_{S/N_j}, this implies the degrees of the two fans are equal, i.e. the number of $(i-1)$-dimensional components in the fans corresponding to N_1 and N_2 are the same. Since J consists of components of dimension $\geq i$, this implies the number of $(i-1)$-dimensional components of F_1 = the number of $(i-1)$-dimensional components of F_2. q.e.d.

Proof of Theorem 6. Let $I_0 \in \mathscr{I}$ and consider the ideal F_0 of its fan. Since saturating with respect to x_{n-1} and x_n removes precisely the points in the corresponding fan; $F_0 = F_L \cap N_0$, where F_L is the ideal of the fan corresponding to L, and has only components of dimensions ≥ 1, and N_0 is an ideal of r points, for some r. Thus we can apply Lemma 3 to conclude that for any ideal $I \in \mathscr{I}$, the corresponding fan has precisely r points.

Let

$$F \equiv F_L \cap N \text{ where } N = \bigcap_{j=1}^{r}(x_0 - c_{0,j}x_n, \cdots, x_{n-1} - c_{n-1,j}x_n),$$

where F_L is as above, and N is the ideal of r distinct points which do not lie on any of the components of F_L.

To see that all such F's lie on the same component, we first need to show that they have the same Hilbert polynomial. This is accomplished by:

Lemma 4. *For any generic choice of c_{i,i_j}'s (where "generic" means none of the points thus defined lie on any component of F_L, and all of the points are distinct) F as defined above has Hilbert polynomial $p(z) = p_{S/F_L} + p_{S/N}$.*

Note. We are assuming all fans corresponding to ideals in $\mathscr{I}$ satisfy this criterion since the definition of Hartshorne's fan requires it.

Proof. Let N be the intersection of the ideals defining the points, so $F_2 = F_L \cap N$. Then

$$0 \to S/F \to S/F_L \oplus S/N \to S/(F_L + N) \to 0$$

is exact, so $p_{S/F} = p_{S/F_L} + p_{S/N} - p_{S/(F_L+N)}$. But the scheme defined by the ideal $F_L + N$ is empty by choice of N (i.e. $F_L + N \supseteq (x_0, \cdots, x_n)^s$ for some s), and so $p_{S/(F_L+N)} = 0$. q.e.d.

Proof of Theorem 6 *continued.* Let $U \subseteq \mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$ be the points of the Hilbert scheme corresponding to ideals F of fans satisfying the hypotheses of Lemma 4. U is an open set of affine N-space, for some suitably large N, and hence U is irreducible. Thus the corresponding fans are parametrized by an irreducible subset of the Hilbert scheme containing, in particular, all fans corresponding to all I's in $\mathscr{I}$. Thus (the points corresponding to) all such fans lie on a single component of the Hilbert scheme.

By Theorem 5 there exists a specialization from the ideal F of the fan corresponding to $I \in \mathscr{I}$ to I itself. Thus *any* component containing F must also contain I. All such F lie on a single component. Therefore all corresponding I's must lie on a single component. This completes the proof of the theorem. q.e.d.

Note. The argument given in Lemma 4 also shows (fairly obviously) that any set of schemes X_1 and X_2 which are given by ideals $I_m = J \cap P_m$ where P_m for each m defines a set of distinct points not intersecting the components defined by J, lie on the same component of the Hilbert scheme.

While the converse to Theorem 6 is in general false, there is one case in which it is true, namely when the component in question is the lexicographic component, i.e. the component of the Hilbert scheme containing the extreme Borel-fixed ideal in the lexicographic order. The relevant definitions and the proof of this converse will be given in Section 5.

Example 1. Here are two examples of partitions.

There are 12 Borel-fixed ideals with Hilbert polynomial $4t + 1$ (the Hilbert polynomial of the rational quartic), but there are only three partitions, defined by the ideals:

1. $\langle a^2, ab, ac, b^2, bc, c^2\rangle$
   (a) $\langle a^2, ab, ac, b^2, bc, c^2\rangle$
2. $\langle a, b^2, bc, c^3\rangle$
   (a) $\langle a^2, ab, ac, ad, b^2, bc, c^3\rangle$
   (b) $\langle a, b^2, bc^2, bcd, c^3\rangle$
   (c) $\langle a, b^2, bc, c^4, c^3d\rangle$
3. $\langle a, b, c^4\rangle$
   (a) $\langle a, b, c^5, c^4d^3\rangle$
   (b) $\langle a, b^2, bc, bd, c^5, c^4d^2\rangle$
   (c) $\langle a^2, ab, ac, ad, b^2, bc, bd, c^5, c^4d\rangle$
   (d) $\langle a, b^2, bc, bd^2, c^5, c^4d\rangle$
   (e) $\langle a^2, ab, ac, ad, b^2, bc, bd^2, c^4\rangle$
   (f) $\langle a, b^2, bc^2, bcd, bd^2, c^4\rangle$
   (g) $\langle a, b^2, bc, bd^3, c^4\rangle$
   (h) $\langle a, b, c^6, c^5d, c^4d^2\rangle$

There are 3865 Borel-fixed ideals with the same Hilbert polynomial as the Veronese surface in $\mathbf{P}^5$, but there are only 12 partitions. The double saturated ideals are the 12 (single saturated) ideals listed above.

5. The radius of the Hilbert scheme

Given $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$, we can construct its *incidence graph*: to each component we assign a vertex, and we connect two vertices if the corresponding components intersect.

Definition. Define the *distance* $d(C, D)$ between two components C, D as the number of edges in the shortest path linking the corresponding vertices. Let $r_D = \max\{d(C, D) | C$ a component of $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}\}$, and define the *radius* of $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$ to be $\min\{r_D | D$ a component of $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}\}$.

Theorem 7. *Let* $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$ *be as above. Let* d (= *degree of* $p(z)$) *be the dimension of the parametrized subschemes. Then the distance from any component to the lexicographic component* (*defined below*) *is* $\leq d+1$, *and thus the radius of* $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$ *is* $\leq d+1$.

The remainder of this section will be devoted to proving this theorem.

Note. In [9], Pardue proves that this theorem also holds when k is an infinite field of characteristic p.

Lemma 5. *Let* $f \in S = k[x_0, \cdots, x_n]$, $f \notin$ *any associated prime of* I. *Then* $\langle f \rangle \cap I = f \cdot I$.

Proof. $(I : f) \cdot f = I \cap \langle f \rangle$. And $(I : f) \cdot f = I \cdot f$, since $(I : f) = I$ by hypothesis. q.e.d.

Proposition 3. *Let* $>$ *be lexicographic order on* $k[x_0, \cdots, x_n]$. *Let* $\{I_{01}, \cdots, I_{0j_0}, I_{11}, \cdots, I_{1j_1}, \cdots, I_{d1}, \cdots, I_{dj_d}\}$ (*where* I_{rs} *defines a subscheme of dimension* r) *be the primes, under a generic change of coordinates, in the fan decomposition of a Borel-fixed ideal* I'. *Let* F *be the corresponding fan. Then*

1. $x_{n-d-1}^{j_d} \cdots x_{n-1}^{j_0}$ *occurs as the last minimal generator of* $\mathrm{in}_>(F)$ *in descending lexicographic order.*
2. *In particular, the fan* F' *corresponding to* $\mathrm{in}_>(F)$ *has the following ideals among those satisfying the conditions of Proposition* 1:

$$(x_0, \cdots, x_{n-d-2}, x_{n-d-1}^{j_d+1}, x_{n-d}^{j_{d-1}+1}, \cdots, x_{n-2}^{j_1+1}, x_{n-1}^{j_0}),$$
$$(x_0, \cdots, x_{n-d-2}, x_{n-d-1}^{j_d+1}, x_{n-d}^{j_{d-1}+1}, \cdots, x_{n-2}^{j_1+1}, x_{n-1}^{j_0-1}),$$
$$\vdots$$
$$(x_0, \cdots, x_{n-d-2}, x_{n-d-1}^{j_d+1}, x_{n-d}^{j_{d-1}+1}, \cdots, x_{n-2}^{j_1+1}, x_{n-1}),$$
$$(x_0, \cdots, x_{n-d-2}, x_{n-d-1}^{j_d+1}, x_{n-d}^{j_{d-1}+1}, \cdots, x_{n-3}^{j_2+1}, x_{n-2}^{j_1}),$$
$$\vdots$$
$$(x_0, \cdots, x_{n-d-2}, x_{n-d-1}^{j_d+1}, x_{n-d}^{j_{d-1}+1}, \cdots, x_{n-3}^{j_2+1}, x_{n-2}),$$
$$\vdots$$
$$(x_0, \cdots, x_{n-d-2}, x_{n-d-1}).$$

Proof (of prop.). By induction on d.

For $d = 0$: The fan decomposition gives j_d distinct points which, under a generic change of coordinates, will be positioned in such a way that each successive projection from the points $(0, \cdots, 0, 1, 0, \cdots, 0)$ will leave j_d distinct points. Hence, projecting onto the line $\mathbf{P}^1$, we get j_d distinct points defined by a single polynomial of degree j_d. In particular, if $I = \bigcap_{s=1}^{j_0} I_{0s}$, then $\mathrm{in}_>(I \cap k[x_{n-1}, x_n]) = \langle x_{n-1}^{j_0} \rangle$. Also, note that, after saturating with respect to x_n, $\mathrm{sat}_{x_n}(\mathrm{in}_>(I)) = (x_0, \cdots, x_{n-2}, x_{n-1}^{j_0}) =: J$. So the fan decomposition of J consists of prime ideals $\{(x_0 - t_0 x_n, \cdots, x_{n-2} - t_{n-2}x_n, x_{n-1} - t_{n-1,i}x_n) | i = j_0, \ldots$ (downto) $1\}$ corresponding to $(x_0, \cdots, x_{n-2}, x_{n-1}^{j_0}), \cdots, (x_0, \cdots, x_{n-2}x_{n-1})$, and $x_{n-1}^{j_0}$ occurs as the last element of J.

Suppose true for $d - 1$, and let the fan F be the intersection of the ideals $\{I_{00}, \cdots, I_{d-1, j_{d-1}}\}$. Let $I_d = \bigcap_{l=1}^{j_d} I_{dl}$. Note that $I_{dl} \cap k[x_{n-1-d}, \cdots, x_n]$ is a single, linear polynomial f_l. Let $I'_{im} = I_{im} \cap k[x_{n-1-d}, \cdots, x_n]$, $I'_d = I_d \cap k[x_{n-1-d}, \cdots, x_n]$. Now, since the fan is reduced (in particular, all components are distinct), and since we are in generic coordinates, we have:

(1) $f_l \neq f_m$ for $l \neq m$, $l, m \in [1, \cdots, j_d]$, and

(2) $(I'_{im} : f_l) = I'_{im}$ for all $i = 0, \cdots, d-1$, $m = 1, \cdots, j_i$ and $l = 1, \cdots, j_d$, since each of these I's is prime and f_l is not in any of them (the components remain distinct under projection), and hence

(3) $(I'_{im} : \prod_{l=1}^{j_d} f_l) = I'_{im}$ for all $i = 0, \cdots, d-1$ and $m = 1, \cdots, j_i$.

By Lemma 5, this implies $I'_{im} \cap I'_d = I'_{im} \cap (\prod_{l=1}^{j_d} f_l) = I'_{im}(\prod_{l=1}^{j_d} f_l)$. Hence $I'_d \cap (F \cap k[x_{n-1-d}, \cdots, x_n]) = I'_d \cdot (F \cap k[x_{n-1-d}, \cdots, x_n])$. In particular,

$$\mathrm{in}_>(I_d \cap F \cap k[x_{n-1-d}, \cdots, x_n]) = x_{n-1-d}^{j_d} \cdot (\mathrm{in}_>(F \cap k[x_{n-1-d}, \cdots, x_n])).$$

Therefore,

$$x_{n-1-d}^{j_d} x_{n-d}^{j_{d-1}} \cdot x_{n-1}^{j_0} \in \mathrm{in}_>(F \cap k[x_{n-1-d}, \cdots, x_n])$$

and remains last by the fact that the order is multiplicative. Thus condition 1 of the proposition holds. Condition 2 now follows from the algorithm to produce fans given in Section 3. q.e.d.

From the proposition, the proof of the following corollary is immediate.

Corollary 1. *Let b_i be the number of components of dimension i in the fan F' defined in the statement of Proposition* 3. *Then $b_i \geq j_i$, the number j of components of dimension i in the fan F, for all $i = 0, \cdots, d$.*

A few more definitions and one more theorem are needed before we can prove the radius theorem:

Definition (Macaulay [8]). Let $J \subseteq k[x_0, \cdots, x_n]_d$ be a monomial ideal generated by monomials of degree d. J is a lexicographic subset of $k[x_0, \cdots, x_n]_d$ if J consists of the r greatest monomials of $k[x_0, \cdots, x_n]_d$ in the lexicographic order for some r. $L \subseteq k[x_0, \cdots, x_n]$ is a lexicographic ideal if L is a monomial ideal, and $(L)_d$ is a lexicographic subset of $k[x_0, \cdots, x_n]_d$ for each d.

Now, given a Hilbert polynomial $p(z)$, we can construct the unique, saturated, lexicographic ideal as follows (see, e.g. Bayer [1]): Given a Hilbert polynomial $p(z)$, there exist integers (Macaulay [8]) $m_0 \geq m_1 \geq \cdots \geq m_d \geq 0$ such that $p(z) = \sum_{i=0}^{r} \binom{z+i}{i+1} - \binom{z+i-m_i}{i+1}$. Let $a_0 = m_r$, $a_1 = m_{r-1} - m_r, \cdots, a_r = m_0 - m_1$. Then

$$L = (x_0, \cdots, x_{n-r-2}, x_{n-r-1}^{a_r+1}, x_{n-r-1}^{a_r} x_{n-r}^{a_{r-1}+1}, \cdots, x_{n-r-1}^{a_r} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1}, \\ x_{n-r-1}^{a_r} \cdots x_{n-1}^{a_0}).$$

Note. Some of the monomials above may not be in a minimal generating set for L. For example, if $a_0 = 0$, then the last monomial, $x_{n-r-1}^{a_r} \cdots x_{n-2}^{a_1}$, divides the monomial preceding it.

Definition. A component of a fan is said to be in *lexicographic position* if the corresponding p_j produced by the algorithm coincides with a p_j produced by the algorithm applied to the lexicographic ideal.

Note. If the dimension of the parametrized subschemes is d, then all dimension $d+1$ and higher components are in lexicographic position trivially.

Theorem 8 (Reeves, Stillman [11]). *On any Hilbert scheme* $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$, *the lexicographic point is a smooth point.*

Definition. The *lexicographic component* of the Hilbert scheme is the component containing the point corresponding to the unique, saturated lexicographic ideal.

Finally, using the above lemmas and propositions, we prove the theorem. During the proof of the theorem, one should keep in mind the following picture: if a Borel-fixed ideal sits in the intersection of two or more components, then the process of producing a fan from such a Borel-fixed ideal will choose the component closest to the lexicographic component by giving us a new ideal whose degenerations include ideals outside the intersection of these components.

Proof of Theorem 7. We shall prove: given a Borel-fixed ideal I with corresponding fan F such that all dimension i and higher components of F are in lexicographic position, there exists a chain of deformations, $I = I_i; I_{i-1}; \dots ; I_0 =$ Lexicographic ideal, deforming I to the lexicographic ideal. This will show that the distance from any component containing I to the lexicographic component is at most i. Since every component contains at least one Borel-fixed ideal, the theorem follows.

We prove this by induction on i. For $i = 0$ this is obvious, since then the only component through the lexicographic point is the lexicographic component (by Theorem 8), so our chain of components consists of the lexicographic component itself.

Suppose the conclusion holds for a Borel-fixed ideal whose corresponding fan has all dimension $i - 1$ and higher components in lexicographic position. Consider I satisfying the above hypothesis, i.e. a Borel-fixed ideal with corresponding fan F having all dimension i and higher components of F in lexicographic position. Let a_j denote the number of j-dimensional components in the fan of I, and let b_j denote the number of j-dimensional components in the fan corresponding to the lexicographic ideal.

By hypothesis, and Lemma 3 applied inductively, starting with dimension $j = d$, $a_j = b_j$ for $j \geq i$. Lemma 3 also implies $a_{i-1} = b_{i-1}$. By the proposition, we know the initial ideal of F under lexicographic order has all these components in lexicographic position, implying that the saturated initial ideal does as well (recall that fans were only defined for saturated monomial ideals). Hence by Lemma 3, the fan of $\mathrm{sat}(\mathrm{in}_{\mathrm{lex.}}(F))$ must have $a_{i-2} = b_{i-2}$. But this implies we are at an ideal which satisfies the above condition for $i-1$. From this new ideal, we are at most $i-1$ steps from the lexicographic ideal. Hence from the ideal we started with, it takes at most i steps to get to the lexicographic ideal. q.e.d.

Corollary 2. *Let a_j denote the number of j-dimensional components in the fan of I, and let b_j denote the number of j-dimensional components in the fan corresponding to the lexicographic ideal. Then $a_j \leq b_j$ for all j.*

Proof. Corollary 1 implies that as we travel towards the lexicographic ideal, the number of components in any dimension never decreases, and the theorem shows that ultimately we arrive at a fan having the same number of components in each dimension as the lexicographic fan. q.e.d.

To prove the next corollary, we need to recall a theorem and a lemma of Bayer and Stillman's:

Theorem 9 (Bayer, Stillman [2]). *Let $I \subseteq S$ be a homogeneous ideal, let $>$ be the reverse lexicographic order, and let $r = \dim(S/I)$. Let*

$$U_r(\mathrm{in}_>(I)) = \{(h_1, \cdots, h_r) \in S_1^r | h_i \text{ is not a zero-divisor on } S/((\mathrm{in}(I), h_1, \cdots, h_{i-1})^{\mathrm{sat}}), 1 \le i \le r\}.$$

If $(x_n, \cdots, x_{n-r+1}) \in U_r(\mathrm{in}_>(I))$, *then* I *and* $\mathrm{in}_>(I)$ *have the same regularity.*

Note. If $\mathrm{in}_>(I)$ is Borel-fixed, then $(x_n, \cdots, x_{n-r+1}) \in U_r(\mathrm{in}_>(I))$ since the associated primes of $\mathrm{in}_>(I)$ in this case are all of the form $(x_1, \cdots, x_j)$ for $1 \le j \le n$.

Lemma 6 (Bayer, Stillman [2]). *Let* $>$ *be the reverse lexicographic order, and choose* i *in the range* $1 \le i \le n$. *Let* $x_n, \cdots, x_{i+1} \in I$, *and let* $m \ge 0$. *Then*

$$(I : x_i)_m = I_m \text{ iff } (\mathrm{in}(I) : x_i)_m = \mathrm{in}(I)_m.$$

We also need the following lemmas to prove the corollary.

Lemma 7. *Let* I *be a saturated ideal in generic coordinates, and let* $>$ *be the reverse lexicographic order. Then* $\mathrm{in}_>(I)$ *is also saturated.*

Proof. I saturated and in generic coordinates implies x_n is not a zero divisor on S/I. Thus $(I : x_n)_m = I_m$ for all $m \ge 0$. By Lemma 6, this implies $(\mathrm{in}_>(I) : x_n)_m = \mathrm{in}_>(I)_m$ for all m. Since $\mathrm{in}_>(I)$ is Borel-fixed, this implies $\mathrm{in}_>(I)$ is saturated, as desired. q.e.d.

Lemma 8. *Let* I *be a saturated, Borel-fixed ideal,* J *the fan derived from* I, *and* $>$ *the lexicographic order. Then the last* (*in pure lexicographic order*) *minimal generator of* $\mathrm{in}_>(J)$ *has degree at least as large as any minimal generator of* I.

Proof. Let $x_0^{b_0} \cdots x_{n-1}^{b_{n-1}}$ be a highest degree minimal generator. Furthermore, if there is more than one such generator, let this be the last which appears in pure lexicographic order. By Theorem 3, this implies there are at least b_{n-1} 0-dimensional components in the fan J. In addition, this implies $x_0^{b_0} \cdots x_{n-2}^{b_{n-2}}$ is in the minimal generating set for $\mathrm{sat}_{x_n, x_{n-1}}(I)$. For if not, then there must be a minimal generator of I of the form $x_0^{c_0} \cdots x_{n-1}^{c_{n-1}}$ with $c_i \le b_i$ for all $i = 1, \cdots, n-2$, and with strict inequality for some i. This monomial comes after the highest degree generator in pure lexicographic order, so it must have lower degree. Using the characterization of Borel-fixed ideals in characteristic 0 (property 1 in Section 2), we see that this implies there exists some monomial in I which actually divides the highest degree generator, a contradiction.

Since $x_0^{b_0} \cdots x_{n-2}^{b_{n-2}}$ appears in the minimal generating set for $\mathrm{sat}_{x_n, x_{n-1}}(I)$, we know by Theorem 3 that the number of components of dimension 1 in the fan J is at least b_{n-2}. Also, the highest degree generator of $\mathrm{sat}_{x_n, x_{n-1}}(I)$ must have degree $\geq \sum_{i=0}^{n-2} b_i$. Repeating this argument for all $i = n-1, n-2, \cdots, 0$, we see that the total number of components in J of all dimensions must be at least $\sum_{i=0}^{n-1} b_i$. By Proposition 3 we know that the last generator of $\mathrm{in}_>(J)$ has degree equal to the sum of the number of components in all dimensions of the fan, and thus its degree must be at least as large as the highest degree generator of J. q.e.d.

Corollary 3 (Gotzmann [5] and Bayer [1]). *The lexicographic ideal has the worst regularity among all saturated ideals on the Hilbert scheme.*

Proof. Given any ideal, we can deform it to a Borel-fixed ideal by applying a generic change of coordinates (which preserves regularity) and then deforming to the initial ideal under some order of the result. Since this initial ideal is a special fiber of a flat family, its regularity must be at least as large as the regularity of the original ideal. Furthermore, if we choose the order to be the reverse lexicographic order, then the resulting initial ideal is also saturated by Lemma 7. Thus the (saturated) Borel-fixed ideal from which we will form the fan for the next stage in the progression towards the lexicographic ideal has regularity at least as large as the regularity of the original ideal. Recall (Section 3) I' is the canonical distraction of I. Let $>$ be reverse lexicographic order, and suppose I is a Borel-fixed ideal.

Claim: $\mathrm{in}_>(I') = I$.

Proof of claim. Recall

$$I' = \left\langle \prod_{i=0}^{n} \prod_{j=1}^{s_i} (x_i - t_{ij}x_n) | x_0^{s_0} \cdots x_{n-1}^{s_{n-1}} \in I_g \right\rangle,$$

where I_g is the minimal generating set of I. Thus the set of minimal generators of I' consists of polynomials H with $\mathrm{in}_>(h) \in I_g$ and $h - \mathrm{in}_>(h) \in \langle x_n \rangle$. So $\mathrm{in}(I') \supseteq I$. But both I' and I have the same Hilbert polynomial, so $\mathrm{in}_>(I') = I$ as desired.

Since $\mathrm{in}_>(I')$ is Borel-fixed, it satisfies the hypotheses of Theorem 9, and hence I' and I have the same regularity. The fans F of Theorem 7 are just canonical distractions in generic coordinates, so the regularity of such fans is the same as the regularity of the Borel-fixed ideals from which they are produced. It follows that each successive Borel-fixed ideal produced in the proof of Theorem 7 (being a specialization of such a fan) must have regularity at least as large as those preceding it and hence at least as large as the original ideal. This is still true after we've saturated the new Borel-fixed ideal, since (1) the last minimal generator of the new Borel-fixed ideal is not divisible by x_n (and thus appears as a minimal generator in the saturation), (2) the last minimal generator has degree greater than any minimal generator in the old Borel-fixed ideal by Lemma 8, and (3) the regularity of a Borel-fixed ideal (in characteristic 0) is just the maximum degree of a minimal generator. Since the lexicographic ideal lies at the end of this chain, it must have the highest regularity of all. q.e.d.

Finally, we prove the converse of Theorem 6 in the case that the component in question is the lexicographic component. To do so, we need to recall some definitions and a theorem of Hartshorne's:

Definition. Let F be a coherent sheaf on $\mathbf{P}_k^n$. Define $R_i(F)$ to be the subsheaf of F whose sections over an open set U are those sections of F over U whose support has codimension $\geq n-i$. For each i, we define

$$n_i(F) = (i!)\,(\text{coefficient of } z^i \text{ in the Hilbert polynomial of } R_i(F)).$$

Also, define $\mathbf{n}_*(\mathbf{F}) = (n_n(F), n_{n-1}(F), \cdots, n_0(F))$.

Note. If F is a fan, then $n_i(F)$ is the number of components of dimension i.

Theorem 10 (Hartshorne, Cor. 2.11 [6]). *Let $Y \subseteq \mathrm{Hilb}_{\mathbf{P}_k^n}^{p(z)}$ be a closed integral subscheme of $\mathrm{Hilb}_{\mathbf{P}_k^n}^{p(z)}$, $X = \mathbf{P}_k^n \times Y$, $f : X \to Y$ projection, and F a flat family on X parametrized by Y. Then $y \mapsto n_*(F_y)$ is an upper semicontinuous function.*

Theorem 11. *Let $L' = \mathrm{sat}_{x_{n-1}, x_n}(L)$, where L is the lexicographic ideal. Then the set*

$$\mathcal{S} = \{I \mid I \subseteq S \text{ is Borel-fixed with Hilbert polynomial } p(z), \quad \text{and } \mathrm{sat}_{x_{n-1}, x_n}(I) = L'\}$$

contains all of the Borel-fixed ideals lying on the lexicographic component of the Hilbert scheme containing L.

Proof. The generic member G of the family of subschemes parametrized by the lexicographic component has $n_*(G) = (b_n, b_{n-1}, \cdots, b_0)$, since it is a union of hypersurfaces of dimension i and degree b_i ($i = 1, \cdots, n$ such that $b_i > 0$) together with b_0 points (see [11] for a proof). By Theorem 10, any ideal I to which G specializes must have $n_*(I) \geq n_*(G)$, and thus by Corollary 1, $n_*(I) = n_*(G)$. The only Borel-fixed ideals satisfying this property are the ones in $\mathcal{S}$. q.e.d.

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