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Murphy's law in algebraic geometry: Badly-behaved deformation spaces

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Abstract. We consider the question: "How bad can the deformation space of an object be?" The answer seems to be: "Unless there is some a priori reason otherwise, the deformation space may be as bad as possible." We show this for a number of important moduli spaces.

More precisely, every singularity of finite type over $\mathbb{Z}$ (up to smooth parameters) appears on: the Hilbert scheme of curves in projective space; and the moduli spaces of smooth projective general-type surfaces (or higher-dimensional varieties), plane curves with nodes and cusps, stable sheaves, isolated threefold singularities, and more. The objects themselves are not pathological, and are in fact as nice as can be: the curves are smooth, the surfaces are automorphism-free and have very ample canonical bundle, the stable sheaves are torsion-free of rank 1, the singularities are normal and Cohen-Macaulay, etc. This justifies Mumford's philosophy that even moduli spaces of well-behaved objects should be arbitrarily bad unless there is an a priori reason otherwise.

Thus one can construct a smooth curve in projective space whose deformation space has any given number of components, each with any given singularity type, with any given non-reduced behavior. Similarly one can give a surface over $\mathbb{F}_p$ that lifts to $\mathbb{Z}/p^7$ but not $\mathbb{Z}/p^8$. (Of course the results hold in the holomorphic category as well.)

It is usually difficult to compute deformation spaces directly from obstruction theories. We circumvent this by relating them to more tractable deformation spaces via smooth morphisms. The essential starting point is

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Mnëv's universality theorem.

The best-laid schemes o' mice an' men
Gang aft agley
An' lea'e us nought but grief an' pain
For promis'd joy!
— Robert Burns, "To a Mouse", 1787

1. Introduction

Define an equivalence relation on pointed schemes generated by: If $(X, p) \to (Y, q)$ is a smooth morphism, then $(X, p) \sim (Y, q)$. We call the equivalence classes *singularity types*, and will call pointed schemes *singularities* (even if the point is regular). We say that *Murphy's law* holds for a moduli space if every singularity type of finite type over $\mathbb{Z}$ appears on that moduli space. Although our methods are algebraic, our arguments all work in the holomorphic category.

1.1. Main Theorem. *The following moduli spaces satisfy Murphy's law.*

M1a. *the Hilbert scheme of nonsingular curves in projective space*
M1b. *the moduli space of maps of smooth curves to projective space (and hence Kontsevich's moduli space of maps)*
M1c. $\mathcal{G}^r_d$ [HM, p. 5], *the space of curves with the data of a linear system of degree d and projective dimension r*
M2a. *the versal deformation spaces of smooth n-folds (with very ample canonical bundle, $n \geq 2$)*
M2b. *the fine moduli stack of smooth n-folds with very ample canonical bundle and reduced automorphism group ($n \geq 2$)*
M2c. *the coarse moduli space of smooth n-folds with very ample canonical bundle ($n \geq 2$)*
*In **M2** we may take the variety X to be simply connected ($\pi_1(X) = 0$ or $\pi_1^{alg}(X) = 0$ depending on the context), with bounded Picard number (in fact Picard number 2), with $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$, and with trivial automorphism group scheme.*
M3. *the Hilbert scheme of nonsingular surfaces in $\mathbb{P}^5$, and the Hilbert scheme of surfaces in $\mathbb{P}^4$*
M4. *the Chow variety of nonsingular curves in projective space, and of nonsingular surfaces in $\mathbb{P}^5$, allowing only seminormal singularities in the definition of Murphy's law (recall that the Chow variety is seminormal* [Kol2, Theorem 3.21]*)*
M5a. *branched covers of $\mathbb{P}^2$ with only simple branching (nodes and cusps), in characteristic not 2 or 3*
M5b. *the "Severi variety" of plane curves with a fixed numbers of nodes and cusps, in characteristic not 2 or 3*

M6. *the moduli space of stable sheaves* [Si]
M7. *the versal deformation spaces of isolated normal Cohen-Macaulay threefold singularities*

The proof are given in the following sections.: **M1** Sect. 5.7, **M2** Sect. 5.8, **M3** Sect. 5.2, **M4–7** Sect. 6.

The meaning of Murphy's law for versal deformation spaces is the obvious one. We should say a few words on why certain moduli spaces exist. **1b:** Although one usually discusses Kontsevich's moduli space of stable maps in characteristic 0, one may as well define the moduli space of maps from nodal curves to projective space, with reduced automorphism group, over Spec $\mathbb{Z}$; this is a Deligne-Mumford stack, essentially by the same construction as that of [FuP]. (It is not proper!) **2b**: [A, p. 182–3] shows existence for surfaces, and the argument applies verbatim in higher dimension. The stack is Deligne-Mumford, locally of finite type. **2c**: [Kol3, Theorem 1.8] shows that there is an algebraic space coarsely representing these moduli functors. (For surfaces, there is even a coarse moduli (algebraic) space of canonical models of surfaces of general type [Kol3, Theorem 1.7].)

1.2. Philosophy. To be explicit about why these results may be surprising: one can construct a smooth curve in projective space whose deformation space has any given number of components, each with any given singularity type, with any given non-reduced behavior along various subsets. Similarly, one can give a smooth surface of general type in characteristic p that lifts to $\mathbb{Z}/p^7$ but not to $\mathbb{Z}/p^8$.

We next give some philosophical comments, which motivated this result. The history sketched in Sect. 2 also provided motivation.

The moral of Theorem 1.1 is as follows. We know that some moduli spaces of interest are "well-behaved" (e.g. equidimensional, having at worst finite quotient singularities, etc.), often because they are constructed as Geometric Invariant Theory quotients of smooth spaces, e.g. the moduli space of curves, the moduli space of vector bundles on a curve, the moduli space of branched covers of $\mathbb{P}^1$ (the Hurwitz scheme, or the space of admissible or twisted covers), the Picard variety, the Hilbert scheme of divisors on projective space, the Severi variety of plane curves with a prescribed number of nodes, the moduli space of abelian varieties (notably [NO]), etc. In other cases, there has been some effort to try to bound how "bad" the singularities can get. Theorem 1.1 in essence states that these spaces can be arbitrarily singular, and gives a means of constructing an example where any given behavior happens.

Murphy's law suggests that unless there is some natural reason for the space to be well-behaved, it will be arbitrarily badly behaved. For example, arithmetically Cohen-Macaulay surfaces in $\mathbb{P}^4$ are always unobstructed [El]; but surfaces in general in $\mathbb{P}^4$ can have arbitrarily bad deformations (by **M3**). Other examples are given in Table 1.

Furthermore, our experience and intuition tells us that pathologies of moduli spaces occur on the boundary, and that moduli spaces of "good"

Table 1.

Well-behaved moduli space	*Badly-behaved moduli space*
curves	surfaces (by **M2b–c**)
branched covers of $\mathbb{P}^1$ (e.g. [HM, Theorem 1.53])	branched covers of $\mathbb{P}^2$ (by **M5a**)
surfaces in $\mathbb{P}^3$	surfaces in $\mathbb{P}^4$ (by **M3**)
Picard variety over the moduli space of curves	its subscheme $\mathcal{G}^r_d$ (by **M1c**)
Severi variety of nodal plane curves (e.g. [HM, Theorem 1.49])	Severi variety of nodal and cuspidal plane curves (by **M5b**)

objects are also "good". Murphy's law shows that this intuition is incorrect; we should expect pathologies even where the objects being parameterized seem harmless. Kodaira says "The theory of deformation was at first an experimental science" [Kod, p. 259]. This result shows that our intuition is flawed because it is based on experimental knowledge of a very small part of the moduli spaces we are interested in; it supports Mumford's philosophy that pathologies are the rule rather than the exception. Alternatively, from the point of view of A. Vershik, this result states that the "universality" philosophy (e.g. [Ve, Sect. 7]) applies widely in algebraic geometry.

As a side comment, Theorem 1.1 indicates that one cannot hope to desingularize the moduli space of surfaces, or any other moduli space satisfying Murphy's law, by adding additional structure; this would imply a resolution of all singularities defined over $\mathbb{Q}$. (Hence the program for desingularization of the space of stable maps informally proposed by some authors seems unlikely to succeed. However, see [VZ] for success in genus 1.)

1.3. Do complex manifolds "care about $\mathbb{Q}$"? To obtain results over other bases, such as algebraically closed fields such as $\mathbb{C}$, note that the spaces above behave well with respect to base change. Hence any singularity obtained by base change from a finite type singularity over $\mathbb{Z}$ may appear.

In most of the above cases, no other singularity may appear. Indeed, any moduli (pseudo-)functor admitting a smooth cover by a scheme locally of finite type over $\mathbb{Z}$ necessarily only has singularities of this sort. For example, the singularity

$$xy(y-x)(y-\pi x)=0 \tag{1}$$

in $\mathbb{C}^2$ may not appear as such a deformation space. Here is a quick sketch of an argument. Consider those singularities consisting of four smooth branches glued together along a divisorial subvariety V, no pair of branches tangent. By considering the fourth-order formal neighborhood of the divisor V in the union, we obtain a λ-invariant which lies in the function field of V. (The kernel of the map $\mathrm{Sym}^4(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^4/\mathfrak{m}^5$ gives an element of $\mathrm{Sym}^4(\mathfrak{m}/\mathfrak{m}^2)$, well-defined up to multiplication by a scalar. This quartic has distinct roots, and hence gives an element λ of $\mathcal{M}_{0,4}/S_4 \cong \mathbb{A}^1$.) This invariant is preserved by smooth pullback. Hence a singularity of type $xy(y-x)(y-mx)=0$ ($m \in \mathbb{C}$) can appear on a scheme of finite type over $\mathbb{Q}$ only if $m \in \overline{\mathbb{Q}}$.

Our motivating question was: how bad can a deformation space be? We have thus answered this question completely in all cases but one. In the case of **M7**, the versal deformation spaces of isolated singularities do not obviously "come from" an Artin stack of finite type over $\mathbb{Q}$.

As the deformation space of a compact complex manifold, with positive canonical bundle, must be "defined over $\mathbb{Q}$" (i.e. appears on a scheme of finite type over $\mathbb{Q}$), no matter how transcendental the defining equations of the surface, we are led naturally to the following question.

1.4. Speculation. *The deformation space of every compact complex manifold is "defined over $\mathbb{Q}$", i.e. is of the same type as a singularity obtained by base change from one of finite type over $\mathbb{Q}$.*

It would be remarkable if the speculation were true: somehow complex geometry would "care" about $\mathbb{Q}$. It would be also remarkable if the speculation were false: then a compact complex manifold would be forced to have this arithmetic property when its canonical bundle were positive, and it may seem unreasonable that positivity might force this arithmetic property.

A vaguer speculation is that all "nice" holomorphic objects have deformation spaces defined over $\mathbb{Q}$. For example: does there exist an isolated complex singularity whose deformation space is equivalent to (1)? What if the singularity is required to be algebraic? Does there exist a compact complex manifold whose deformation space has such singularity type? What if the manifold is required to be projective? It would be very interesting to have *any* example of a non-pathological object (e.g. isolated complex algebraic singularity, complex projective manifold, or even non-algebraic examples) with deformation space not equivalent to one of finite type over $\mathbb{Q}$.

Hence it would be very interesting to have a proof of the speculation, or a single counterexample. We suspect that the speculation is false, as for each of the moduli problems that are Artin stacks of finite type (over $\mathbb{Q}$), the construction requires some kind of boundedness argument, coming from some positivity. (This is related in spirit to Belyi's theorem.) Nonetheless, it is not clear where one might find a counterexample.

1.5. *Notation.* Let Def denote the versal or Kuranishi deformation space (not the space of first-order deformations). The object being deformed will be clear from the context.

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2. History, and further questions

2.1. Hilbert schemes. The motivation for both the equivalence relation $\sim$ and the terminology "Murphy's law" comes from the folklore conjecture that the Hilbert scheme "satisfies Murphy's law".

2.2. Law [HM, p. 18]**.** *There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme.*

I am not sure of the origin of this philosophy, but it seems reasonable to ascribe it to Mumford. This traditional statement of Murphy's law is admittedly informal and imprecise (see the MathReview [Lax]). Clearly not every singularity appears on the Hilbert scheme of projective space. For example, the only zero-dimensional Hilbert schemes (of projective space) are reduced points. Allowing "smooth equivalence classes" of singularities seems the mildest way of rescuing the law.

In his famous paper [Mu], Mumford described a component of the Hilbert scheme of space curves that is everywhere nonreduced. Other examples of nonreduced components of the Hilbert scheme have since been given [GP,Kl,E,M-DP]. Other pathologies relating to the number of components of the Hilbert scheme of *smooth* space curves were given by Ellia, Hirschowitz, and Mezzetti [EHM], and by Fantechi and Pardini [FP1]. (The results of the latter will be essential to our argument.)

Raynaud's example (see Sect. 2.3) gives a component of a Hilbert scheme of smooth surfaces which exists in characteristic p, but does not lift to characteristic 0 (by the standard methods of Lemma 5.1). Mohan Kumar, Peterson, and Rao [MPR] give a component of the Hilbert scheme of smooth surfaces in $\mathbb{P}^4$ which exists in characteristic 2 but does not lift. See [EHa, Sect. 3] for more on problems of lifting curves out of characteristic p.

Although the Hilbert scheme of projective spaces was suspected to behave badly, other moduli spaces were believed (or hoped) to be better-behaved. We now discuss these.

2.3. Surfaces and higher-dimensional varieties. (See [Ca2] for an excellent overview of the subject.) The first example of an obstructed smooth variety was due to Mumford, obtained by blowing up his curve in $\mathbb{P}^3$ [Mu, pp. 643–644]. The first example of an obstructed surface is due to Kas [Kas].

Other examples were later given by Burns and Wahl [BW], and later many others. Horikawa [Ho], Miranda [Mi], and Catanese [Ca3] gave examples of generically nonreduced components of the moduli space of surfaces; in each case the surfaces did not have ample canonical bundle, and this appeared to be a common explanation of this pathology [Ca3, p. 294]. (Although the examples of Catanese are surfaces of general type with nonreduced deformation spaces, their canonical models have smooth deformation spaces [Ca3, Prop. 1.14].) Catanese conjectured that if S is a surface of general type with $q = 0$ and K_S ample, then the moduli space $\mathfrak{M}(S)$ is smooth on an open dense set ([Ca2, p. 34, 69], [Ca3, p. 294]). Theorem 1.1 **M2b–c** gives a counterexample to this conjecture, and as Catanese pointed out, even to the stronger conjecture where K_S is very ample. Manetti gave an earlier counterexample in his thesis [Man1, Corollary 3.4]; the added advantage of **M2** is that every (finite type) nonreduced structure is shown to occur.

V. Alexeev has recently suggested that the corrected hope should be that the moduli space of surfaces is well-behaved when the canonical bundle of the surfaces are "barely positive". It would be very interesting to make this statement precise.

Catanese showed that the moduli space of complex surfaces in a given homeomorphism class can have arbitrarily many components of different dimension [Ca1, Theorem A], and asked if this were still true for those in a given diffeomorphism class [Ca1, p. 485]; Theorem 1.1 **M2b–c** answers this in the affirmative. A prior answer was recently given by Catanese and B. Wajnryb [CaW]. The added benefit of **M2** is that all possibilities are shown to occur.

Serre gave the first example of a projective variety that could not be lifted to characteristic 0 [Ser]. Raynaud gave the first example of such a surface [Ray]; W. Lang gave more [Lang]. R. Easton has used ideas related to this paper to produce counterexamples to the Bogomolov–Miyaoka–Yau inequality in positive characteristic [Ea].

2.4. Plane curves with nodes and cusps. If C is a reduced complex plane curve, the classical question of "completeness of the characteristic linear series" asks (in modern language) if an appropriate equisingular moduli space is smooth. Severi proved this is true if C has only nodes ([Sev], see also [Z, Sect. VIII.4]), and asserted this if C has nodes and cusps [W3]. (See [Z, pp. 116–117 and Sect. VIII] for motivation for the study of nodal and cuspidal plane curves.) It was later realized that Severi's assertion was unjustified. Enriques tried repeatedly to show that such curves were unobstructed [Ca2, p. 51]; Zariski also raised this question [Z, p. 221]. The first counterexample was given by Wahl [W1, Sect. 3.6], and another was given by Luengo [Lu]. Theorem 1.1 **M5b** shows that Severi was in some sense "maximally wrong".

2.5. Stable coherent sheaves. The moduli space of stable coherent sheaves is due to Simpson [Si]. Our example is in fact a torsion-free sheaf on $\mathbb{P}^5$;

the theory of the moduli of torsion-free sheaves was developed earlier by Maruyama [Mar], building on Gieseker's work in the surface case [Gi].

2.6. Singularities. The theory of deformations of singularities is too large to summarize here. We point out however that it was already established by Burns and Wahl [BW] that such deformation spaces can be bad, although not this pathological.

2.7. Further questions. Theorem 1.1, and the philosophy and history given above, beg further questions. Do deformations of surface singularities (say isolated and Cohen-Macaulay) satisfy Murphy's law? How about the Hilbert scheme of curves in $\mathbb{P}^3$? The Hilbert scheme of points on a smooth threefold? The moduli of vector bundles on smooth surfaces? Can the extra dimensions allowed in the definition of *type* be excised, i.e. can "smooth" be replaced by "étale" in the definition of type? (As observed above, this is not possible for the Hilbert scheme.) Catanese asks if Murphy's law for surfaces is still true if we require not only that the surface has very ample canonical bundle, but also that the canonical embedding is cut out by quadrics. *Conjecture:* for any given p, the surfaces whose canonical divisor induces an embedding satisfying property N_p satisfy Murphy's law. The case $p = 1$ is Catanese's question. One might hope that for any nonsingular variety, a sufficiently positive nonsingular divisor has this property; this would imply that the conjecture is true, using the construction of Sect. 5.8.

3. The starting point: Mnëv's universality theorem

We will prove Theorem 1.1 by drawing connections among various moduli spaces, taking as a starting point a remarkable result of Mnëv. Define an *incidence scheme of points and lines in* $\mathbb{P}^2$, a locally closed subscheme of $(\mathbb{P}^2)^m \times (\mathbb{P}^{2*})^n = \{p_1, \ldots, p_m, l_1, \ldots, l_n\}$ parameterizing $m \geq 4$ marked points and n marked lines, as follows.

- $p_1 = [1; 0; 0]$, $p_2 = [0; 1; 0]$, $p_3 = [0; 0; 1]$, $p_4 = [1; 1; 1]$.
- We are given some specified incidences: For each pair (p_i, l_j), either p_i is required to lie on l_j, or p_i is required not to lie on l_j.
- The marked points are required to be distinct, and the marked lines are required to be distinct.
- Given any two marked lines, there a marked point required to be on both of them.
- Each marked line contains at least three marked points.

3.1. Theorem (Mnëv). *Every singularity type of finite type over $\mathbb{Z}$ appears on some incidence scheme.*

This is a special case of Mnëv's Universality Theorem [Mn1,Mn2]. A short proof is given by Lafforgue in [Laf, Théorème 1.14]. Lafforgue's construction does not necessarily satisfy the first, fourth and fifth requirements of an incidence scheme, but they can be satisfied by adding more points. (The only subtlety in adding these extra points is verifying that in the configuration constructed by Lafforgue, no three lines pass through the same point unless required to by the construction.) *Caution:* Other expositions of Mnëv's theorem do not prove the result scheme-theoretically, only "variety-theoretically," as this is all that is needed for most purposes.

For the rest of the paper fix a singularity type of finite type over $\mathbb{Z}$. Our goal will be to find this singularity type on each of the spaces given in Theorem 1.1. By Mnëv's Theorem 3.1, there is an incidence scheme exhibiting this singularity type at a certain configuration $\{p_1, \dots, p_m, l_1, \dots, l_n\}$. Consider the surface S that is the blow-up of $\mathbb{P}^2$ at the points p_i. Let C be the proper transform of the union of the l_j, so C is a smooth curve (a union of $\mathbb{P}^1$'s). This induces a morphism from the incidence scheme to the moduli space of surfaces with marked smooth divisors.

3.2. Proposition. *This morphism is étale at* $(\mathbb{P}^2, \{p_i\}, \{l_j\}) \mapsto (S, C)$.

Thus the singularity at $(\mathbb{P}^2, \{p_i\}, \{l_j\})$ has the same type as the moduli space of surfaces with marked smooth divisor at (S, C).

Proof. We will produce an étale-local inverse near (S, C). Consider a deformation of (S, C):

$$\begin{tikzcd} (S, C) \arrow[r, hook] \arrow[d] & (\mathcal{S}, \mathcal{C}) \arrow[d] \\ \mathrm{pt} \arrow[r, hook] & B. \end{tikzcd} \tag{2}$$

Pull back to an étale neighborhood of pt so that the components of C are labeled. The Hilbert scheme of (-1)-curves is étale over the base [GrFGA, Sect. 5].

Let E_i be the (-1)-curve corresponding to p_i. Pull back to an étale neighborhood so that the points of the Hilbert scheme corresponding to E_i extend to sections (so there are divisors $\mathcal{E}_i$ on the total space of the family that are (-1)-curves on the fibers). By abuse of notation, we use the same notation (2) for the resulting family. By Castelnuovo's criterion, $\mathcal{S}$ can be blown down along the $\mathcal{E}_i$ so that the resulting surface is smooth, with marked sections extending $\{p_i\}$. (Castelnuovo's criterion over an Artin local scheme follows from general results of J. Wahl. Suppose we have a smooth surface $\tilde{X}$ over Artin local scheme $\operatorname{Spec} A$ with closed point $\operatorname{Spec} k$, such that $X := \tilde{X} \times_A k$ contains a (-1)-curve E, and $\pi : X \to Y$ is the blow-down. We seek a smooth $\tilde{Y} \to \operatorname{Spec} A$, and $\tilde{X} \to \tilde{Y}$ extending $X \to Y$. By shrinking Y if necessary, we may assume Y is affine and $\pi(E)$ is cut out by 2 equations in $H^0(Y, \mathcal{O}_Y)$. [W2, Thm. 1.4(b)] states that the desired $\tilde{X} \to \tilde{Y} \to \operatorname{Spec} A$ exists if $H^2(X, \mathcal{O}_X) = 0$ and $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is

A-flat. But $H^2(X, \mathcal{O}_X) = 0$ as X may be covered by 2 affine open sets, and $H^1(X, \mathcal{O}_X) = H^0(Y, R^1\pi_*\mathcal{O}_X) = H^0(Y, 0) = 0$, and by [W2, Cor. 0.4.2] or direct induction, $H^1(X, \mathcal{O}_X) = 0$ implies $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$, as desired. We note that Wahl's results are independent of char k. The flat morphism $\tilde{Y} \to \operatorname{Spec} A$ is then smooth because Y is smooth over k. For arguments in the holomorphic category, see [Ho] and [KolM, Proposition 11.4.2].)

The central fiber is then $\mathbb{P}^2$, so (as $\mathbb{P}^2$ is rigid) the family is locally trivial. The marked points $p_1, \dots, p_4$ give a canonical isomorphism with $\mathbb{P}^2$. (We may need to restrict to a smaller neighborhood to ensure that these points are in general position.) As the components $\{C_j\}$ of C necessarily meet various E_i, their images $\{l_j\}$ necessarily pass through the necessary p_i. □

4. From abelian covers to Murphy's law for surfaces

We use this intermediate moduli space of surfaces with marked divisors to prove **M2**, by connecting such marked surfaces to abelian covers. We use the theory of abelian covers developed by Catanese, Pardini, Fantechi, and Manetti [Ca1,P,FP1,Man2]. (Bidouble covers were introduced by Catanese. Pardini developed the general theory of abelian covers. Key deformation-theoretic results were established by Fantechi-Pardini and Manetti.) Let $G = (\mathbb{Z}/p)^3$, where $p = 2$ or 3 is prime to the characteristic of the residue field of the singularity. Let $G^\vee$ be the dual group, or equivalently the group of characters. Let $\langle \cdot, \cdot \rangle : G \times G^\vee \to \mathbb{Z}/p$ be the pairing (after choice of root of unity ζ), which we extend to $\langle \cdot, \cdot \rangle : G \times G^\vee \to \mathbb{Z}$ by requiring $\langle \sigma, \chi \rangle \in \{0, \dots, p-1\}$. Suppose we have two maps $D : G \to \operatorname{Div}(S)$, $L : G^\vee \to \operatorname{Pic}(S)$. We say (D, L) satisfies the *cover condition* [P, Proposition 2.1] if (D, L) satisfies $D_0 = 0$ and

$$pL_\chi = \sum_\sigma \langle \sigma, \chi \rangle D_\sigma$$

for all σ, χ. (Equality is taken in $\operatorname{Pic}(S)$.)

4.1. Proposition (Pardini)**.** *Suppose (D, L) satisfies the cover condition, and suppose the D_σ are nonsingular curves, no three meeting in a point, such that if D_σ and $D_{\sigma'}$ meet then they are transverse and σ and σ' are linearly independent in G. Then:*

(i) *There is a corresponding G-cover $\pi : \tilde{S} \to S$ with branch divisor $D = \cup D_\sigma$.*
(ii) *$\tilde{S}$ is nonsingular.*
(iii) $\pi_*\mathcal{O}_{\tilde{S}} = \oplus_\chi \mathcal{O}_S(-L_\chi)$.
(iv) $\pi_*\mathcal{K}_{\tilde{S}} \cong \oplus_\chi \mathcal{K}_S(L_\chi)$. *The Galois group G acts on the left side in the obvious way; it acts on the χ-summand on the right by the character χ.*

Note for future reference that the branch divisor D_σ corresponds to the subgroup of G generated by σ. (Note also that (iii) and (iv) are consistent with Serre duality on $\tilde{S}$.)

Proof. (i) is [P, Proposition 2.1], (ii) is [P, Proposition 3.1], and (iii) is a consequence of Pardini's construction [P, (1.1)]. Pardini points out that (iv) is a special case of duality for finite flat morphisms, see [Ha] Exercises III.6.10 and Ex. III.7.2. (It also follows by a straightforward local calculation. See [Ca1, p. 495] for the analogous proof for bidouble covers. The generalization to abelian covers is analogous to Pardini's proof of (iii).) □

The next two examples apply to (S, C) produced at the end of Sect. 3. If the character of the residue field is 2 (respectively 3), then only Example 4.3 (respectively 4.2) applies; otherwise both apply.

4.2. *Key example:* $p = 2$. Fix $\sigma_0 \neq 0$ in G. Let A be a sufficiently ample bundle such that $A \equiv C \pmod 2$. Let $D_{\sigma_0} = C$, $D_0 = 0$, and let D_σ be a general section of A otherwise, such that $D_{\sigma'}$ meets $D_{\sigma''}$ transversely for all $\sigma' \neq \sigma''$. Let $L_0 = 0$, $L_\chi = 2A$ if $\langle \sigma_0, \chi \rangle = 0$ and $\chi \neq 0$, and $L_\chi = (3A + C)/2$ if $\langle \sigma_0, \chi \rangle = 1$. (As $\operatorname{Pic} S$ is torsion-free, there is no ambiguity in the phrase $(3A + C)/2$.) It is straightforward to verify that (D, L) satisfies the hypotheses of Proposition 4.1.

4.3. *Key example:* $p = 3$. Fix $\sigma_0 \neq 0$ in G, and $\chi_0 \in G^\vee$ such that $\langle \sigma_0, \chi_0 \rangle = 1$. Let A be a sufficiently ample bundle such that $A \equiv C \pmod 3$. Let $D_{\sigma_0} = C$, D_σ be a general section of A if $\langle \sigma, \chi_0 \rangle = 1$ and $\sigma \neq \sigma_0$, and $D_\sigma = 0$ otherwise. Let

- $L_\chi = (8A + C)/3$ if $\langle \sigma_0, \chi \rangle = 1$
- $L_0 = 0$
- $L_\chi = 3A$ if $\langle \sigma_0, \chi \rangle = 0$ and $\chi \neq 0$
- $L_{-\chi_0} = (16A + 2C)/3$
- $L_\chi = (7A + 2C)/3$ if $\langle \sigma_0, \chi \rangle = 2$ and $\chi \neq -\chi_0$

It is straightforward to verify that (D, L) satisfies the hypotheses of Proposition 4.1 (note that if $\sigma \neq 0$, then at most one of $\{D_\sigma, D_{-\sigma}\}$ is nonzero).

4.4. Theorem. *In Examples 4.2 and 4.3, if A is sufficiently ample, then:*

(a) *$K_{\tilde{S}}$ is ample. In particular, $\tilde{S}$ is of general type, and is its own canonical model.*
(b) *$\tilde{S}$ is regular: $q(\tilde{S}) := h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$.*
(c) *The deformations of $\tilde{S}$ are the same as the deformations of $(S, \{D_\sigma\})$. In particular, the deformations of G-covers are also G-covers.*
(d) *The deformation space of $\tilde{S}$ has the same type as the deformation space of (S, C).*
(e) *$\tilde{S}$ has no infinitesimal automorphisms.*

Part (d) implies that the fine moduli stack of surfaces of general type satisfies Murphy's law. Of course (e) is immediate in characteristic 0, as $\tilde{S}$ has ample canonical bundle, hence $h^0(\tilde{S}, T_{\tilde{S}}) = h^2(\tilde{S}, \Omega_{\tilde{S}}(K_{\tilde{S}})) = 0$ by Kodaira vanishing and Serre duality.

We will not need this fact, but it is true that by choosing A sufficiently positive, one may show that $K_{\tilde{S}}$ is *very* ample. I am grateful to F. Catanese for pointing this out. The argument (directly generalizing Catanese's argument [Ca1, p. 502] for bidouble covers) is given in an earlier version of this paper [V, Theorem 4.4].

Proof. (a)

$$2K_{\tilde{S}} = \pi^* \left(2K_S + \sum D_\sigma\right) = \pi^*(2K_S + C + qA)$$

where $q = 6$ if $p = 2$ and $q = 8$ if $p = 3$. If A is sufficiently ample, then $2K_S + \sum D_\sigma$ is ample, hence (as π is finite) $K_{\tilde{S}}$ is ample.

(b) By the Leray spectral sequence,

$$h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = h^1(S, \pi_* \mathcal{O}_{\tilde{S}}) = \sum_\chi h^1\big(S, L_\chi^{-1}\big) = 0$$

using Serre vanishing (for $\chi \neq 0$) and the regularity of any blow-up of $\mathbb{P}^2$ (for $\chi = 0$).

(c) For example 4.2 ($p = 2$), the result follows from [Man2, Corollary 3.23]; we restate the three hypotheses of Manetti's result for the reader's convenience. (i) S is smooth of dimension ≥ 2, and $H^0(S, T_S) = 0$. (The latter is true because S has no non-trivial infinitesimal automorphism. Reason: any such would descend to an infinitesimal automorphism of $\mathbb{P}^2$ fixing the p_i, in particular $p_1 = [1;0;0], \dots, p_4 = [1;1;1]$.) (ii) $H^0(S, T_S(-L_\chi)) = \operatorname{Ext}^1_{\mathcal{O}_S}(\Omega^1_S, L_\chi^{-1}) = H^1(S, L_\chi^{-1}) = 0$ (true by Serre vanishing, and sufficient ampleness of A). (iii) $H^0(S, D_\sigma - L_\chi) = 0$ for all $\chi \neq 0$, $\langle \sigma, \chi \rangle = 0$ (true by Serre vanishing). Hence (c) holds for Example 4.2.

The paper [Man2] deals with $(\mathbb{Z}/2)^r$ covers. However, [Man2, Corollary 3.23] applies without change for $(\mathbb{Z}/p)^r$-covers. The only change in the proof arises in the proof of the prior result [Man2, Proposition 3.16]; the statement of this Proposition remains the same, and the proof is changed in the obvious way. In particular, the fourth equation display should read

$$\Omega^1_{X/Y} = \bigoplus_\sigma \frac{\mathcal{O}_X(-(p-1)R_\sigma)}{w_\sigma \mathcal{O}_X(-pR_\sigma)} = \bigoplus_\sigma \mathcal{O}_{R_\sigma}(-(p-1)R_\sigma).$$

Then the hypotheses of [Man2, Corollary 3.23] follow as in the case $p = 2$, and we have proved (c) for Example 4.3 ($p = 3$) as well.

(d) Choose $A = C + npK_{\tilde{S}}$ for $n \gg 0$, so that its higher cohomology vanishes. Then $\operatorname{Def}(S, \{D_\sigma\}) \to \operatorname{Def}(S, C)$ is a smooth morphism: in any deformation of S the divisor class $[D_\sigma]$ extends (as C and $K_{\tilde{S}}$ extend), and extends uniquely (by $h^1(S, \mathcal{O}_S) = 0$), and the choice of divisor in the divisor class is a smooth choice.

(e) Note that

$$h^0\left(\tilde{S}, T_{\tilde{S}}\right) = h^0\left(S, \pi_* T_{\tilde{S}}\right) = \sum_{\chi} h^0\left(S, \pi_* T_{\tilde{S}}\right)^{\chi}$$

where the sum is the character decomposition. By [P, Proposition 4.1(a)],

$$h^0\left(S, \pi_* T_{\tilde{S}}\right)^{\chi} = h^0\Big(S, TS\Big(-\log \sum_{\sigma:\langle\sigma,\chi\rangle \neq p-1} D_\sigma\Big) \otimes L_\chi^{-1}\Big)$$

which is 0 by sufficient ampleness of A. (The case $\chi = 0$ should be considered slightly differently.) $\square$

At this point we have already proved **M2a** and **M2b** for surfaces, except that our surfaces have automorphisms, are not obviously simply connected, and have high Picard number. If we are willing to ignore these requirements, then we can prove **M2c** for surfaces as well, by showing that the automorphism group scheme of $\tilde{S}$ is precisely G (so the isotropy group of the moduli functor is constant near $[\tilde{S}]$), as follows. By Theorem 4.4(e), $\tilde{S}$ has no nontrivial infinitesimal automorphisms. Then Fantechi and Pardini's [FP1, Theorem 4.6] shows that $\tilde{S}$ has only $|G|$ (noninfinitesimal) automorphisms.

By taking the product of $\tilde{S}$ with general curves of sufficiently high genus, we obtain an n-fold with deformation space of the same singularity type; this argument is described in an earlier version of this paper [V, Sect. 5]. This proves **M2**, minus the requirements of simple connectedness etc.

5. Relating deformation spaces

We now describe three techniques that will relate deformation spaces smoothly.

5.1. Lemma. *Let X be a regular variety ($h^1(X, \mathcal{O}_X) = 0$) with a map $X \to \mathbb{P}^n$, such that $h^1(X, \mathcal{O}_X(1)) = 0$. Suppose either (i) $h^2(X, \mathcal{O}_X) = 0$, or (ii) $\mathcal{O}_X(1)$ is a $\mathbb{Q}$-multiple of K_X. Then* $\operatorname{Def}(X \to \mathbb{P}^n) \to \operatorname{Def} X$ *is smooth.*

Proof. Choose a basis for $h^0(\mathbb{P}^n, \mathcal{O}(1))$, and let $s_0, \ldots, s_n$ be the restriction of the basis to X. Then $\operatorname{Def}(X, \mathcal{O}_X(1); s_0, \ldots, s_n \in H^0(X, \mathcal{O}_X(1)))$ $\to \operatorname{Def}(X \to \mathbb{P}^n)$ is smooth (of relative dimension 1). Furthermore, $\operatorname{Def}(X, \mathcal{O}_X(1); s_0, \ldots, s_n \in H^0(X, \mathcal{O}_X(1))) \to \operatorname{Def}(X, \mathcal{O}_X(1))$ is smooth by comparing the deformation-obstruction theories of the two functors:

$$\begin{aligned}\longrightarrow H^0\big(X, \mathcal{O}_X(1)\big)^{\oplus n+1} &\longrightarrow \operatorname{def}\big(X, \mathcal{O}_X(1); s_0, \dots, s_n \in H^0(X, \mathcal{O}_X(1))\big)\\ &\longrightarrow \operatorname{def}\big(X, \mathcal{O}_X(1)\big)\end{aligned}$$

$$\begin{aligned}\longrightarrow H^1\big(X, \mathcal{O}_X(1)\big)^{\oplus n+1} &\longrightarrow \operatorname{ob}\big(X, \mathcal{O}_X(1); s_0, \dots, s_n \in H^0(X, \mathcal{O}_X(1))\big)\\ &\longrightarrow \operatorname{ob}(X, \mathcal{O}_X(1)).\end{aligned}$$

Here def denotes first-order deformations, and ob denotes obstructions. Finally, to show that $\operatorname{Def}(X, \mathcal{O}_X(1)) \to \operatorname{Def}(X)$ is an isomorphism, note that over any infinitesimal deformation of X, $\mathcal{O}_X(1)$ deforms by (i) or (ii), and deforms uniquely as $h^1(X, \mathcal{O}_X) = 0$. □

5.2. *Application: Proof of* **M3.** By applying Lemma 5.1 to an embedding $\tilde{S} \to \mathbb{P}^n = \mathbb{P}^5$ by a sufficiently positive multiple of $K_{\tilde{S}}$ (using Theorem 4.4(b)), we see that the Hilbert scheme of nonsingular surfaces in $\mathbb{P}^5$ satisfies Murphy's law. Using $n = 4$ instead yields a surface in $\mathbb{P}^4$ with singularities in codimension 2; each consists of two nonsingular branches meeting transversely. The deformations of such a singularity preserve the singularity. (This can be checked formally locally; the calculation can then be done using two transverse co-ordinate planes in $\mathbb{A}^4$, which is [Ha, Exercise 9.9].) Hence deformations of the singular surface in $\mathbb{P}^4$ correspond to deformations of the nonsingular surface $\tilde{S}$ along with the map to $\mathbb{P}^4$, concluding the proof of **M3**.

5.3. Deformations of blow-ups of projective space.

5.4. Theorem. *Let $f : X = \mathrm{Bl}_Z \mathbb{P}^N \to \mathbb{P}^N$ be the blow-up of $\mathbb{P}^N$ along a nonsingular subvariety Z, over a field. Then* $\operatorname{Def}(f : X \to \mathbb{P}^N) \to \operatorname{Def} X$ *is an isomorphism.*

The definition of $\operatorname{Def}(f : X \to Y)$ is the obvious one, see for example [Ran, Definition 1.1]. This result, in much more generality, is certainly known, but we were unable to find a precise statement in the literature, so we have contented ourselves with the straightforward special case we will use. In the smooth holomorphic case, the result is very similar to Horikawa's [Ho, Theorem 8.2].

Note that [Ran, Theorem 3.3] states that if $f : X \to Y$ is a morphism with $f_* \mathcal{O}_X = \mathcal{O}_Y$ and $R^1 f_* \mathcal{O}_X = R^2 f_* \mathcal{O}_X = 0$, then $\operatorname{Def}(f : X \to Y) \to \operatorname{Def}(X)$ is smooth. The proof seems to not need $R^2 f_* \mathcal{O}_X = 0$, and seems to give the stronger conclusion that $\operatorname{Def}(f : X \to Y) \to \operatorname{Def}(X)$ is an isomorphism (see the e-print version of this paper). Thus Theorem 5.4 would follow. However, the referee notes that the proof in [Ran] is not complete, since for Example [Ran, (6)] is not justified, as unless f is flat, the hypotheses for the Grothendieck spectral sequence are not satisfied.

Proof. Suppose $\tilde{X}$ is a deformation of X, i.e. we have a flat morphism $\pi : \tilde{X} \to \operatorname{Spec} A$ to a local Artinian scheme with closed point $\operatorname{Spec} k$, and an isomorphism $\tilde{X} \times_A k \cong X$. As $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$, the invertible sheaf $\mathcal{O}_X(1) := f^*\mathcal{O}_{\mathbb{P}^N}(1)$ extends uniquely to an invertible sheaf $\mathcal{L}$ on $\tilde{X}$. As $R^1 f_* \mathcal{O}_X = 0$ and the natural morphism $\mathcal{O}_{\mathbb{P}^N} \to f_*\mathcal{O}_X$ is an isomorphism, we have that $R^1 f_* \mathcal{O}_X(1) = 0$ and $\mathcal{O}_{\mathbb{P}^N}(1) \to f_*\mathcal{O}_X(1)$ is an isomorphism. Then the Leray spectral sequence for f implies that (i) $h^1(X, \mathcal{O}_X(1)) = 0$ and (ii) the map $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(X, \mathcal{O}_X(1))$ is an isomorphism. Thus $R^1\pi_*\mathcal{L} = 0$, so as π is flat, $\pi_*\mathcal{L}$ is locally free (= free), of rank $h^0(X, \mathcal{O}_X(1)) = N + 1$. (Here we are using cohomology and base change, see for example [W2] Theorem 0.4 and corollaries.) Thus for each deformation $\tilde{X}$, we get a unique $f : \tilde{X} \to \mathbb{P}^N_A$, up to automorphisms of $\mathbb{P}^N_A$ fixing $\mathbb{P}^N_k$. □

5.5. Fantechi and Pardini's slicing trick. Our third smoothness criterion is due to Fantechi and Pardini. If $X \subset \mathbb{P}^n$ is a subscheme, let $\operatorname{Hilb}(X)$ be the (connected component of the) Hilbert scheme containing $[X]$.

5.6. Theorem. *(a) (Fantechi-Pardini* [FP2, Proposition 4.2]*) Let $V \subset \mathbb{P}^n$ be a smooth, regular, projectively normal variety. Let H be a smooth hypersurface of degree l in $\mathbb{P}^n$ meeting V transversely along W, and let $U \subset \operatorname{Hilb}(V) \times \operatorname{Hilb}(H)$ be the open set of pairs (V', H') such that V' and H' are smooth and transverse and V' is projectively normal. If $l \gg 0$, then the morphism $U \to \operatorname{Hilb}(W)$ (induced by the intersection) is smooth.*

(b) Furthermore, W is embedded by a complete linear system.

Fantechi and Pardini's proof of (a) invokes Kodaira vanishing to show that if F is a hypersurface of degree l then $H^1(F, N_{F/\mathbb{P}^n}) = 0$, but this may be easily checked directly, so their result is not characteristic-dependent.

Proof of (b). If $\mathcal{I}_{W/V}$ is the ideal sheaf of W in V, we have the exact sequence

$$0 \longrightarrow \mathcal{I}_{W/V}(1) \longrightarrow \mathcal{O}_V(1) \longrightarrow \mathcal{O}_W(1) \longrightarrow 0.$$

As $\mathcal{I}_{W/V} \cong \mathcal{O}_V(-l)$, $h^1(V, \mathcal{I}_{W/V}(1)) = 0$ by Serre vanishing (as $l \gg 0$). Thus $H^0(V, \mathcal{O}_V(1)) \to H^0(W, \mathcal{O}_W(1))$ is surjective. As $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(V, \mathcal{O}_V(1))$ is also surjective (V is embedded by a complete linear system), the result follows. □

5.7. *Application: Proof of* **M1**. By Lemma 5.1, embed $\tilde{S}$ in $\mathbb{P}^n$ by the complete linear system of a sufficiently large multiple of $K_{\tilde{S}}$, so $\operatorname{Def}(\tilde{S} \hookrightarrow \mathbb{P}^n) \to \operatorname{Def}(\tilde{S})$ is smooth. Theorem 5.6(a) (with $V = \tilde{S}$) then gives **M1a**: the curve in question is the intersection of $\tilde{S}$ with a general hypersurface of sufficiently high degree. Deformations of a smooth curve in $\mathbb{P}^n$ are the same as deformations of the corresponding immersion, yielding **M1b**. Theorem 5.6(b) gives **M1c**.

5.8. *Proof of* **M2.** We now prove **M2** for d-folds ($d > 1$). Our strategy is a variation of Horikawa's, and a similar strategy was used in Manetti's thesis [Man1, Sect. V.3]. Fix a singularity type, and choose a surface $\tilde{S}$ as produced in Sect. 4 whose deformation space has that singularity type. Use a sufficiently positive multiple of $K_{\tilde{S}}$ to embed $\tilde{S}$ in $\mathbb{P}^N$ ($N \geq d + 2$). By Theorem 5.4, $\operatorname{Def}(\operatorname{Bl}_{\tilde{S}} \mathbb{P}^N)$ has the same type as $\operatorname{Def}(\tilde{S} \hookrightarrow \mathbb{P}^N)$, which by Lemma 5.1 has the same type as $\operatorname{Def} \tilde{S}$. Note that $\operatorname{Bl}_{\tilde{S}} \mathbb{P}^N$ is

(*) simply connected, with Picard number 2, and with $h^i(\mathcal{O}) = 0$ for $0 < i < \dim$.

Use a sufficiently positive bundle to embed $\operatorname{Bl}_{\tilde{S}} \mathbb{P}^N$ in $\mathbb{P}^{N'}$, and slice with a hypersurface of sufficiently large degree to obtain $W \hookrightarrow \mathbb{P}^{N'}$ ($\dim W > d$, $h^1(W, \mathcal{O}_W(1)) = 0$). By the Lefschetz hyperplane theorem, W satisfies (*) as well. By Theorem 5.6, $\operatorname{Def}(W \hookrightarrow \mathbb{P}^{N'})$ has the same type as $\operatorname{Def}(\operatorname{Bl}_{\tilde{S}} \mathbb{P}^N \hookrightarrow \mathbb{P}^{N'})$, which in turn has the same type as $\operatorname{Def}(\operatorname{Bl}_{\tilde{S}} \mathbb{P}^N)$ (by Lemma 5.1) and hence $\operatorname{Def} \tilde{S}$. Also by Lemma 5.1, $\operatorname{Def} W$ has the same type as $\operatorname{Def}(W \hookrightarrow \mathbb{P}^{N'})$ and hence $\operatorname{Def} \tilde{S}$. Furthermore, W has ample canonical bundle. Use a complete linear system for a sufficiently positive multiple of K_W to embed W in $\mathbb{P}^{N''}$ (so that $h^1(W, \mathcal{O}_W(1)) = 0$).

Repeatedly slice W with hypersurfaces of sufficiently large degree, to obtain a $(d + 1)$-fold $X' \hookrightarrow \mathbb{P}^{N''}$. (We will use X' in the proof of Proposition 5.9 below.) Slice once more to obtain a d-fold $X \hookrightarrow \mathbb{P}^{N''}$. By the Lefschetz hyperplane theorem, X also satisfies (*), and (by a short induction, using $d > 1$) $h^1(X, \mathcal{O}_X(1)) = 0$. By repeatedly using Theorem 5.6, $\operatorname{Def}(X \hookrightarrow \mathbb{P}^{N''})$ has the same type as $\operatorname{Def}(W \hookrightarrow \mathbb{P}^{N''})$, hence $\operatorname{Def} W$ (Lemma 5.1), and hence $\operatorname{Def} \tilde{S}$. By Lemma 5.1 again, $\operatorname{Def} X$ has the same type as $\operatorname{Def}(X \hookrightarrow \mathbb{P}^{N''})$. (We use the fact that W is pluricanonically embedded in Lemma 5.1 in the case $d = 2$.) We have thus proved **M2a**. The proof of **M2b–c** is now completed by the following result.

5.9. Proposition. *X has trivial automorphism group scheme.*

Proof. We first show that X has no infinitesimal automorphisms. (In characteristic 0, this is clear, as K_X is very ample; see the remark after the statement of Theorem 4.4.) We have shown that $\tilde{S}$ has no infinitesimal automorphisms (Theorem 4.4(e)), and we construct X from $\tilde{S}$ by repeatedly using three constructions, so we show that the desired property behaves well with respect to these constructions. Let aut denote the space of infinitesimal automorphisms. As $\operatorname{aut} \tilde{S} = 0$ and $\tilde{S}$ is embedded non-degenerately in projective space $\mathbb{P}^N$, we have $\operatorname{aut}(\tilde{S} \hookrightarrow \mathbb{P}^N) = 0$. As $\operatorname{aut}(\tilde{S} \hookrightarrow \mathbb{P}^N) = 0$, $\operatorname{aut}(\operatorname{Bl}_{\tilde{S}} \mathbb{P}^N) = 0$ as well: any tangent field of the blow-up descends to a tangent field on $\mathbb{P}^N$, which fixes $\tilde{S}$. If Z is a sufficiently positive nonsingular divisor on a nonsingular Y, then $\operatorname{aut} Y = 0$ implies $\operatorname{aut} Z = 0$, from the long exact sequence for $0 \to T_Y(-Z) \to T_Y(-\log Z) \to T_Z \to 0$, using

$h^1(Y, T_Y(-Z)) = 0$ and $h^0(Y, T_Y(-\log Z)) \leq h^0(Y, T_Y) = 0$. Hence we have shown the rigidity of the following objects, in order:

$$\tilde{S} \implies \tilde{S} \hookrightarrow \mathbb{P}^N \implies \mathrm{Bl}_{\tilde{S}}\, \mathbb{P}^N \implies W \implies X' \implies X.$$

We now show that X has no noninfinitesimal automorphisms. As X' has ample canonical bundle and no infinitesimal automorphisms, it has finite automorphism group. Recall that X is the intersection of X' with a hypersurface of sufficiently high degree. As X is embedded $\mathbb{Q}$-canonically in $\mathbb{P}^{N''}$ by a complete linear system, and $\operatorname{Pic} X \cong \mathbb{Z}^2$ is torsion-free, the automorphisms of X are in bijection with automorphisms of $\mathbb{P}^{N''}$ fixing X (as a set). We will show that the only $(N''+1)\times(N''+1)$ matrices fixing X (as a set) are multiples of the identity. Over the space M of $(N''+1)\times(N''+1)$ matrices *not* fixing X', consider the intersection $X' \cap \phi(X') \subset M \times \mathbb{P}^{N''}$ (where ϕ is the universal matrix over M). The fibers of $X' \cap \phi(X') \to M$ each have dimension at most $d = \dim X' - 1$. Choose the degree of X to be bigger than the degree of any d-dimensional fiber of $X' \cap \phi(X') \to M$ (which is a semicontinuous function on M). Then X cannot be fixed by any matrix in M.

Hence any automorphism of X arises from one of the (finite number of) automorphisms of X'. Choose a point of X' on which $\operatorname{Aut} X'$ acts faithfully; any hypersurface vanishing at only one point of the orbit is necessarily fixed only by the identity automorphism of X'. Thus a general hypersurface X on X' inherits only the trivial automorphism from X'. □

6. From surfaces to the rest of Theorem 1.1

6.1. *Proof of* **M4.** Near a seminormal point of the Hilbert scheme, there is a morphism from the Hilbert scheme to the Chow variety [Kol2, Theorem 6.3]. If the point of the Hilbert scheme parametrizes an object that is geometrically reduced, normal, and of pure dimension, then this morphism is a local isomorphism [Kol2, Corollary 6.6.1]. Hence **M4** follows from **M1a** and **M3**.

6.2. *Proof of* **M5.** **M5a** follows from Lemma 5.1, by taking three sections of a sufficiently positive multiple of $K_{\tilde{S}}$ on $\tilde{S}$. J. Wahl provides the connection to **M5b**:

6.3. Theorem (Wahl [W1, p. 530])**.** *Let $Y \to \mathbb{P}^2$ be a finite surjective morphism, Y a nonsingular surface, whose branch curve C is reduced with only nodes and cusps as singularities. Then via taking branch curves, there is a one-to-one correspondence between infinitesimal deformations of the morphism $Y \to \mathbb{P}^2$ and infinitesimal deformations of C in $\mathbb{P}^2$ which preserve the formal nature of the singularities.*

Wahl's paper assumes that the characteristic is 0, but his proof of this result uses only that the characteristic is not 2 or 3. To reassure the reader, we point out the places where characteristic 0 is used before Wahl's proof of Theorem 6.3 concludes on p. 558. Proposition 1.3.1 and equation (1.5.3) are not used in the proof. Theorem 2.2.8 and its rephrasing (Theorem 2.2.11) give a normal form for stable singularities, and use only that the characteristic is not 2 or 3. (One might conjecture that an appropriate formulation is true in characteristic 2 and 3, but I have not attempted to prove this.) Part **M5b** then follows from the next result.

6.4. Proposition. *If $\tilde{S}$ is any smooth projective surface over an infinite base field of characteristic not 2 or 3, and $\mathcal{L}'$ is an ample invertible sheaf, then for $n \gg 0$, three general sections of ${\mathcal{L}'}^{\otimes n}$ give a morphism to $\mathbb{P}^2$ with reduced branch curve with only nodes and cusps as singularities.*

The result is tedious and relatively straightforward to prove, and the proof is omitted. (However, it is given in the first e-print version of this paper [V, Proposition 6.2].) In characteristic 0 the result is classical (presumably nineteenth century); the proof is by taking n large enough that ${\mathcal{L}'}^{\otimes n}$ is very ample, and then taking a generic projection. Because we need the result in positive characteristic as well, a slightly different approach is necessary, although as usual we show the result by showing that "nothing worse can happen," by excluding possibilities on a case-by-case basis.

6.5. *Proof of* **M6.** (I am grateful to E. Markman and R. Thomas for discussions.) The sheaf in question will be the ideal sheaf $\mathcal{I}$ of the image of $\tilde{S}$ in $\mathbb{P}^4$ (from **M3**). The next result implies **M6**.

6.6. Proposition. *If Y is a nonsingular variety with $h^1(Y, \mathcal{O}_Y) = 0$, and $X \hookrightarrow Y$ is a subscheme of codimension at least 2, then the deformation space of $X \hookrightarrow Y$ is canonically isomorphic to the deformation space of the ideal sheaf $\mathcal{I}$ of X (as a torsion-free sheaf).*

Proof. (The central observation here is due to Kollár.) We have the obvious morphism $\operatorname{Def}(X \hookrightarrow Y) \to \operatorname{Def} \mathcal{I}$. We describe the morphism in the other direction. Let $\mathcal{J}$ be the universal torsion-free sheaf over $\operatorname{Def} \mathcal{I}$ (a sheaf on $Y \times \operatorname{Def} \mathcal{I}$). The reflexive hull $\mathcal{J}^{**}$ is an invertible sheaf by [Koll, Lemma 6.13], and thus a deformation of the structure sheaf over the central fiber. As $h^1(Y, \mathcal{O}_Y) = 0$, there are no nontrivial deformations of the structure sheaf, so $\mathcal{J}^{**} \cong \mathcal{O}_{Y \times \operatorname{Def} \mathcal{I}}$. The canonical morphism $\mathcal{J} \to \mathcal{J}^{**} \cong \mathcal{O}_{Y \times \operatorname{Def} \mathcal{I}}$ is an inclusion, as $\mathcal{J}$ is torsion-free (by flatness, and torsion-freeness over the central fiber). Thus $\mathcal{J}$ is an ideal sheaf. Let $\mathcal{Q}$ be the quotient $\mathcal{O}_{Y \times \operatorname{Def} \mathcal{I}} / \mathcal{J}$. As the restriction $\mathcal{I}$ of $\mathcal{J}$ to the central fiber is torsion-free, the restriction of $\mathcal{J} \to \mathcal{O}_{Y \times \operatorname{Def} \mathcal{I}}$ to the central fiber remains injective, hence $\mathcal{Q}$ is flat over $\operatorname{Def} \mathcal{I}$. (We use here the local criterion for flatness [Ei, Theorem 6.8, Exercise 6.5]. Let $(R, \mathfrak{m})$ be the complete local ring such that $\operatorname{Spf} R = \operatorname{Def} \mathcal{I}$. We check R-flatness of $\mathcal{Q}$ by verifying that $\operatorname{Tor}_1^R(R/\mathfrak{m}, \mathcal{Q}) = 0$. As $\mathcal{O}_{Y \times \operatorname{Def} \mathcal{I}}$ is R-flat, this is equivalent to showing that the restriction of $\mathcal{J} \to \mathcal{O}_{Y \times \operatorname{Def} \mathcal{I}}$ to the central fiber remains injective.) Thus we have described a morphism $\operatorname{Def} \mathcal{I} \to \operatorname{Def}(X \hookrightarrow Y)$. By following the universal families under both morphisms, we see that the two morphisms are inverse to each other. □

6.7. *Proof of* **M7**. We obtain the threefold singularity by embedding $\tilde{S}$ in projective space by a complete linear system arising from a sufficiently positive multiple of $K_{\tilde{S}}$ (Lemma 5.1 again). The deformations of the cone over the surface are the same as the deformations of the surface in projective space, by the following theorem of Schlessinger.

6.8. Theorem (Schlessinger [Sch, Theorem 2]). *Let $\tilde{S} \subset \mathbb{P}^n$ be a projectively normal variety (over a field) of dimension ≥ 2, such that*

$$h^1\left(\tilde{S}, \mathcal{O}_{\tilde{S}}(v)\right) = h^1\left(\tilde{S}, T_{\tilde{S}}(v)\right) = 0$$

for $v > 0$. Then the versal deformation spaces of $\tilde{S}$ in $\mathbb{P}^n$ and the singularity $C_{\tilde{S}}$ (the cone over $\tilde{S}$) are isomorphic.

(Although Schlessinger works in the complex analytic category, his proof is purely algebraic, and characteristic-independent.) This singularity is Cohen-Macaulay by the following result, concluding the proof of **M7**.

6.9. Proposition. *Suppose $\tilde{S}$ is a Cohen-Macaulay scheme (over a field), $h^i(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$ for $i = 1, \ldots, \dim \tilde{S} - 1$ and $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 1$. Then the embedding of $\tilde{S}$ by a sufficiently ample line bundle is arithmetically Cohen-Macaulay.*

This result follows from a statement of Hartshorne and Ogus [HaO, p. 429 #3]. See [GW, pp. 207–208] or [CuH, Lemma 1.1(2)] for a proof. The hypotheses follow from the regularity of $\tilde{S}$, Theorem 4.4(b). (It turns out that in characteristic 0, $2K_{\tilde{S}}$ is ample enough, using Kodaira vanishing.)

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