Math 311 HW SET 2 SOLUTIONS

2.1.2

Suppose $\{a_n\}$, $\{b_n\}$ are bounded below. Then for all n

$A_1 \le a_n \le A_2$ for some A_1, A_2 and

$B_1 \le b_n \le B_2$ for some B_1, B_2.

(i) Let $A = \max\{|A_1|, |A_2|\}$, $B = \max\{|B_1|, |B_2|\}$. Then

$0 \le |a_n| \le A$ and $0 \le |b_n| \le B$ so that

$a_n b_n \le |a_n||b_n| \le AB$ for all n, so $\{a_n b_n\}$ is bounded above.

(ii) Alternatively, suppose one of the sequences has a limit $L > 0$. Without loss of generality let $\lim_{n\to\infty} a_n = L > 0$. Then for sufficiently large n

$\cancel{a_n b_n \le a_n |b_n| \le L|b_n|}$ so

either $b_n \le 0$ in which case $a_n b_n \le 0$ or

$B > b_n > 0$ in which case $a_n b_n \le LB$.

In both cases $\{a_n b_n\}$ is bounded above for large n.

Since the first N terms are bounded by $\max\{a_n b_n \mid n \le N\}$

then $\{a_n b_n\}$ is bounded above.

Note: (i) can be weakened to the case $a_n > 0$, $b_n > 0$. This is (still) sufficient. (ii) can be strengthened to the case a_n is eventually positive for n large.

2.2.1

(a) $-1 \le \cos n \le 1$ so $2 \le 3 + \cos n \le 4$ and so $\frac{1}{4} \le \frac{1}{3+\cos n} \le \frac{1}{2}$.

(b) Inspection suggests an upper bound (sharpest) at $n=1$ and a lower bound at $n=4$ (again, sharpest). Indeed, for $n \ge 2$

$$-\frac{1}{n^2+1} \le \frac{\sin n}{n^2+1} \le \frac{1}{n^2+1} \le \frac{1}{5} < \frac{\sin(1)}{2} \approx .421$$ and for $n \ge 5$

and $$-.045 \approx \frac{\sin(4)}{17} < -\frac{1}{26} \approx -.039 \le -\frac{1}{n^2+1} \le \frac{\sin(n)}{n^2+1} \le \frac{1}{n^2+1}$$

so that these are infact bounds for $\frac{\sin(n)}{n^2+1}$.

2.4.2

Assume $|a_i| \le 1$ for all a_i, then

$$|a_1 \sin(b) + a_2 \sin(2b) + \dots + a_n \sin(nb)| \le |a_1 \sin(b)| + |a_2 \sin(2b)| + \dots + |a_n \sin(nb)|$$ (by extended triangle inequality)

$$\le |a_1| + |a_2| + \dots + |a_n|$$

$$\le 1 + 1 + \dots + 1$$

$$= n$$

So by contraposition, if $|a_1 \sin(b) + \dots + a_n \sin(nb)| > n$, then $|a_i| > 1$.

.6.1 Choose $N = a$ so that $n > N \Rightarrow$

$$\frac{a^{n+1}}{(n+1)!} = \frac{a^n}{n!} \cdot \frac{a}{(n+1)} < \frac{a^n}{n!}$$ since $\frac{a^n}{n!} > 0$ and $\frac{a}{n+1} < 1$.

Then the sequence is eventually monotonic (decreasing).

2.6.2 Let $b_n = \frac{n}{a^n}$.

Consider
$$b_{n+1} - b_n = \frac{n+1}{a^{n+1}} - \frac{n}{a^n}$$
$$= \frac{n+1-na}{a^{n+1}}$$
$$= -\frac{n(a-1)-1}{a^{n+1}}$$

Since $a > 1$ then $a^{n+1} > 0$, and if $n > \frac{1}{a-1}$, $n(a-1)-1 > 0$.

Let $N = \frac{1}{a-1}$, then for any $n > N$, $b_{n+1} - b_n < 0$ ie $b_{n+1} < b_n$.

So, the sequence $b_n = \frac{n}{a^n}$ is monotonic (decreasing) for n large.

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roblem 2-1

(a) $a_n \le a_{n+1}$ so $n \cdot a_{n+1} \ge a_1 + \dots + a_n$,

so $n(a_1 + \dots + a_n) + n \cdot a_{n+1} \ge n(a_1 + \dots + a_n) + (a_1 + \dots + a_n)$

and
$$b_{n+1} = \frac{a_1 + \dots + a_n + a_{n+1}}{n+1} \ge \frac{a_1 + \dots + a_n}{n} = b_n$$

So b_n is increasing.

(b) $a_n \le A$ for some A and all n. Then $b_n = \frac{a_1 + \dots + a_n}{n} \le \frac{A + \dots + A}{n} = \frac{nA}{n} = A$ for all n. Hence b_n is bounded above.

2-2: Since $\{a_n\}$ is bounded and increasing, then by Completeness Property, $\{a_n\}$ has a limit L. Note that $a_n \le L$.

Then given $\varepsilon = \frac{1}{2}$, there exists an integer $N > 0$ s.t. for any $n \ge N$.

$0 \le L - a_n \le \frac{1}{2}$.

Since $L - a_n = (L - a_{n+1}) + (a_{n+1} - a_n) \ge a_{n+1} - a_n$, then $0 \le a_{n+1} - a_n \le \frac{1}{2}$ for any $n \ge N$. Since $\{a_n\}$ is an integer sequence, then $a_{n+1} - a_n = 0$ for any $n \ge N$.