Math 311 Solutions to HW 3

3.1.1 a) $$\left|\frac{\sin n - \cos n}{n} - 0\right| \leq \left|\frac{\sin n}{n}\right| + \left|\frac{\cos n}{n}\right| \quad \text{(By the triangle inequality)}$$

$$\leq \left|\frac{1}{n}\right| + \left|\frac{1}{n}\right| \quad (\text{Since } |\sin n| \leq 1,\ |\cos n| \leq 1)$$

$$= \frac{2}{n}$$

Since $\frac{2}{n}$ can be made arbitrarily small $\lim_{n\to\infty} \frac{\sin n - \cos n}{n}$ must be 0. Specifically this is the case for n large enough that $\frac{2}{n} < \varepsilon$ or $\frac{2}{\varepsilon} < n$. Then given $\varepsilon > 0$ let $N = \frac{2}{\varepsilon}$. Now $n > N \Rightarrow \left|\frac{\sin n - \cos n}{n} - 0\right| < \varepsilon$ which is the desired result.

d) $$\left|\frac{n}{n^3-1}\right| < \left|\frac{n}{n^3-n}\right| = \frac{1}{n^2-1}$$

Given some $\varepsilon > 0$, $\frac{1}{n^2-1} < \varepsilon \Leftrightarrow n > \sqrt{\frac{1}{\varepsilon}+1}$.

Then let $N = \sqrt{\frac{1}{\varepsilon}+1}$, so $n > N$ gives

$\left|\frac{n}{n^3-1} - 0\right| < \varepsilon$. Thus $\lim_{n\to\infty} \frac{n}{n^3-1} = 0$.

e) $$\left|\sqrt{n^2+2} - n - 0\right| = \frac{2}{\sqrt{n^2+2}+n} < \frac{2}{n}.$$ The rest is identical to (a).

3.2.2 $a_n < b_n < a_{n+1} < \cdots < L$

so that $a_n - L < b_n - L < a_{n+1} - L < \cdots < 0$
Given $\varepsilon > 0$ let n be sufficiently large enough that $|a_n - L| < \varepsilon$, then in fact

$$-\varepsilon < a_n - L < b_n - L < \cdots < 0 < \varepsilon$$

So that $|b_n - L| < \varepsilon$ as desired.
Thus $\lim_{n \to \infty} b_n = L$.

3.2.4 $$\frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n} < \underbrace{\frac{1}{n^2} + \cdots + \frac{1}{n^2}}_{n} = \frac{n}{n^2} = \frac{1}{n}$$

Then let $N = \frac{1}{\varepsilon}$ so $n > N \Rightarrow \frac{1}{n} < \varepsilon$

and thus $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n^2+k} = 0$.

3.3.2. By definition, given some $M > 0$ there exists $N > 0$ such that $n > N \Rightarrow a_n > M$.
Also ~~given any~~ there exists some N_0 such that $n > N_0 \Rightarrow b_n > a_n$. Let $N_1 = \max\{N, N_0\}$ so that $n > N_1 \Rightarrow$

$$b_n > a_n > M.$$

Since M is arbitrary, b_n tends to infinity.

3.6.1 a) Natural log, on the interval $[1,2]$, attains its maximum value at 2, i.e., is bounded above by $\ln 2 < 1$. Then

$$\int_1^2 (\ln x)^n \, dx < \int_1^2 (\ln 2)^n \, dx = (\ln 2)^n$$

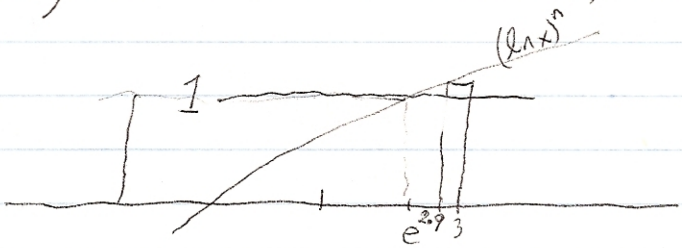
Now $\lim_{n\to\infty} (\ln 2)^n = 0$ since $(\ln 2)^n < \varepsilon$ whenever

$n \ln \ln 2 < \ln \varepsilon$ whenever

$n > \ln\varepsilon (\ln \ln 2)^{-1}$

hence $\lim_{n\to\infty} \int_1^2 (\ln x)^n \, dx = 0$

b) $\ln x$ is positive, and increasing on $[2,3]$. From the following diagram it should be clear that the integral is bounded below by $.1(\ln(2.9))^n$



But for any given $N>0$, $.1(\ln(2.9))^n > N$

when $n \ln \ln 2.9 > N$

equivalently $n > N(\ln \ln 2.9)^{-1}$

So therefore $\lim_{n\to\infty}\int_2^3 \ln^n x \, dx = \infty$

Problem 3-1

a) By definition given $\varepsilon > 0$ $\exists N$ s.t. $n > N \Rightarrow |a_n| < \varepsilon$

Then write

$$b_n = \frac{a_1 + \cdots + a_{\lceil N \rceil}}{n} + \frac{a_{\lceil N \rceil + 1} + \cdots + a_n}{n}$$

where $\lceil N \rceil$ is the smallest integer greater or equal to N. Let $M = \left| \frac{a_1 + \cdots + a_{\lceil N \rceil}}{\varepsilon} \right|$ so

that $n > M \Rightarrow \varepsilon > \left| \frac{a_1 + \cdots + a_{\lceil N \rceil}}{n} \right|$. Then

$n > \max\{N, M\} \Rightarrow$

$$b_n = \left| \frac{a_1 + \cdots + a_{\lceil N \rceil}}{n} \right| + \left| \frac{a_{\lceil N \rceil + 1} + \cdots + a_n}{n} \right|$$

$$\leq \varepsilon + \frac{|a_{\lceil N \rceil + 1}| + \cdots + |a_n|}{n}$$ by extended triangle inequality

$$\leq \varepsilon + \frac{n\varepsilon}{n} = 2\varepsilon$$

By K-ε principle $\lim_{n \to \infty} b_n = 0$

b) $$\frac{a_1 - L + a_2 - L + \cdots + a_n - L}{n} = \frac{a_1 + a_2 + \cdots + a_n}{n} - L$$

So that $a_n \to L$ iff $a_n - L \to 0$ iff $b_n - L \to 0$ (by part (a)) iff $b_n \to L$.

problem

3.4 Let $\lim_{n\to\infty} a_n = L$ and let $\varepsilon > 0$. Let N be such that $n > N \Rightarrow L - \varepsilon < a_n < L + \varepsilon$.

For the first $\lfloor N \rfloor$ terms we have upper and lower bounds $A_1 = \max\{a_n \mid 1 \le n \le \lfloor N \rfloor\}$ and $A_2 = \min\{a_n \mid 1 \le n \le \lfloor N \rfloor\}$. Then $\max\{A_1, L+\varepsilon\}$ and $\min\{A_2, L-\varepsilon\}$ are upper and lower bounds for a_n (respectively). Our choice of ε was arbitrary. Take $\varepsilon = 1$ if a specific bound is desired.