Math 311 HW 4 Solutions.

Problem 4-1

Let $e_n = n^{\frac{1}{n}} - 1$ be the error term. Note that $n > 1 \Rightarrow n^{1/n} > 1 \Rightarrow e_n > 0$. From the binomial theorem,

$$n = (1+e_n)^n = 1 + ne_n + \tfrac{1}{2}n(n-1)e_n^2 + \cdots + e_n^n.$$

Since each term is positive, $n > \frac{1}{2}n(n-1)e_n^2$ so that $e_n < \sqrt{\frac{2}{n-1}}$. Then $\lim_{n\to\infty} e_n = 0$, therefore $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Exercise 5.1.1.

a) $$\lim_{n\to\infty} \frac{n^3 - n^2 - 1}{2n^3 + 1} = \lim_{n\to\infty} \frac{1 - \frac{1}{n} - \frac{1}{n^3}}{2 + \frac{1}{n^3}} = \frac{\lim_{n\to\infty} 1 - \frac{1}{n} - \frac{1}{n^3}}{\lim_{n\to\infty} 2 + \frac{1}{n^3}}$$

by the quotient theorem. Now $\lim_{n\to\infty} \frac{1}{n^k} \le \lim_{n\to\infty} \frac{1}{n} = 0$ for $k \ge 1$ (by the squeeze theorem) so that, by linearity,

$$\frac{\lim_{n\to\infty} 1 - \frac{1}{n} - \frac{1}{n^3}}{\lim_{n\to\infty} 2 + \frac{1}{n^3}} = \frac{1-0-0}{2+0} = \frac{1}{2}.$$

b) $$\lim_{n\to\infty} \frac{(n-3)}{(2n-1)} 2^{-n} = \left(\lim_{n\to\infty} \frac{n-3}{2n-1}\right) \cdot \left(\lim_{n\to\infty} 2^{-n}\right)$$

by the product theorem, since both limits exist.

$\lim_{n\to\infty} \frac{n-3}{2n-1} = \frac{1}{2}$ by the same reasoning as in a)

and $\lim_{n\to\infty} 2^{-n} = 0$ (theorem 3.4) so

$$\lim_{n\to\infty} \frac{n-3}{2n-1} 2^{-n} = \frac{1}{2} \cdot 0 = 0.$$

Exercise 5.1.4

We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n}{b_n}\cdot b_n = \left(\lim_{n\to\infty}\frac{a_n}{b_n}\right)\cdot\left(\lim_{n\to\infty} b_n\right)$ (by the product theorem)

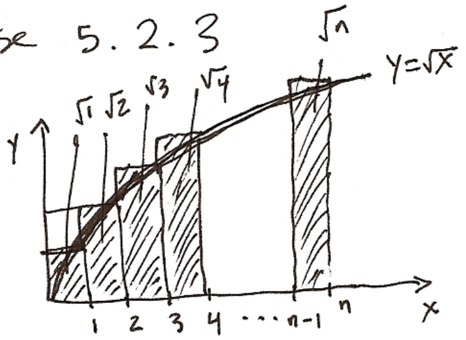
$= L\cdot 0 = 0.$

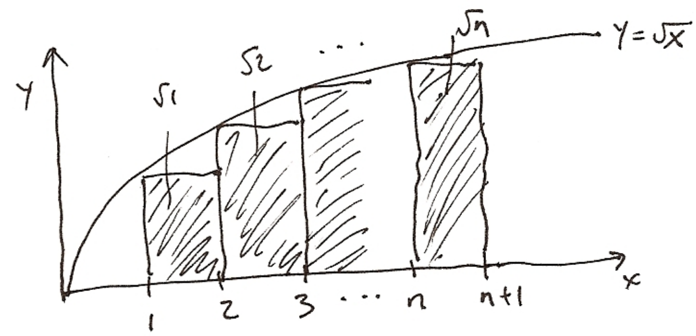
Exercise 5.2.1

For all n, $\frac{\sqrt{n}-1}{\sqrt{n+1}} \le \frac{\sqrt{n}+\cos n}{\sqrt{n+1}} \le \frac{\sqrt{n}+1}{\sqrt{n+1}}$.

Now, $\lim_{n\to\infty}\frac{\sqrt{n}-1}{\sqrt{n+1}} = \lim_{n\to\infty}\frac{1-\frac{1}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}}} = 1$ (by quotient theorem). Also $\lim_{n\to\infty}\frac{\sqrt{n}+1}{\sqrt{n+1}} = \lim_{n\to\infty}\frac{1+\frac{1}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}}} = 1$ so therefore, by the squeeze theorem, $\lim_{n\to\infty}\frac{\sqrt{n}+\cos n}{\sqrt{n+1}} = 1$.

Exercise 5.2.3





From the above two graphs we see

$$\int_0^n \sqrt{x}\,dx \le \sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} \le \int_1^{n+1}\sqrt{x}\,dx$$

i.e., $\frac{2}{3}n^{3/2} \le a_n \le \frac{2}{3}(n+1)^{3/2} - \frac{2}{3}$.

Then $1 \le \frac{a_n}{\frac{2}{3}n^{3/2}} \le \frac{\frac{2}{3}(n+1)^{3/2}-\frac{2}{3}}{\frac{2}{3}n^{3/2}} = \left(\frac{n+1}{n}\right)^{3/2} - \frac{1}{n^{3/2}}$.

Since $1 \le \left(\frac{n+1}{n}\right)^{3/2} = \left(1+\frac{1}{n}\right)^{3/2} \le \left(1+\frac{1}{n}\right)^2$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 1$ by thm 5.1 then by squeeze theorem $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{3/2} = 1$ thus $\lim_{n\to\infty}\frac{a_n}{\frac{2}{3}n^{3/2}} = 1$,

hence $a_n \sim \frac{2}{3}n^{3/2}$.