Math 311 HW 6 Solutions

6.4.2, 6-1, 7.1.1, 7.2.2, 7.2.4

6.4.2 $|a_n - a_{n+1}| < CK^n$ for $n \gg 1$

Then $|a_n - a_m| \leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots |a_{m-1} - a_m|$

$\leq CK^n + CK^{n+1} + \cdots + CK^{m-n+1} < C\sum_{j=n}^{\infty} K^j$

$= C\left(\left(\frac{1}{1-K}\right) - \left(\frac{1-K^n}{1-K}\right)\right) = C\frac{K^n}{1-K}$

which is proportional to K^n and so approaches 0 as $n \to \infty$. Hence a_n is Cauchy.

6-1 (a) If $X_n = \dfrac{X_{n-1} + X_{n-2}}{2}$ then

$$X_n - X_{n-1} = \frac{X_{n-2} - X_{n-1}}{2} = \left(-\tfrac{1}{2}\right)(X_{n-1} - X_{n-2})$$

So $X_n - X_m = \cancel{\left(\tfrac{-1}{2}\right)}\ X_n - X_{n-1} + X_{n-1} - X_{n-2} + \cdots + X_{m+1} - X_m$

$$= \left[\left(-\tfrac{1}{2}\right)^{n-m} + \left(-\tfrac{1}{2}\right)^{n-m-1} + \cdots + \left(-\tfrac{1}{2}\right) + 1\right](X_{m+1} - X_m)$$

$$= \left(\sum_{k=0}^{n-m}\left(-\tfrac{1}{2}\right)^k\right)\left(-\tfrac{1}{2}\right)^m (x_1 - x_0)$$

$$< \left(\sum_{k=0}^{\infty}\left(-\tfrac{1}{2}\right)^k\right)\left(-\tfrac{1}{2}\right)^m (x_1 - x_0)$$

$$= \tfrac{2}{3}\left(-\tfrac{1}{2}\right)^m (x_1 - x_0)$$

So that $X_n - X_m \to 0$ when $n > m \gg 1$.

(b) Let $Y_n = \dfrac{X_n - X_0}{X_1 - X_0}$. Then $Y_0 = 0$, $Y_1 = 1$

and $Y_n = \dfrac{Y_{n-1} + Y_{n-2}}{2} = \sum_{k=0}^{n-1}\left(-\tfrac{1}{2}\right)^k$.

Then $\lim_{n\to\infty} Y_n = \sum_{k=0}^{\infty}\left(-\tfrac{1}{2}\right)^k = \tfrac{2}{3}$ and $\lim_{n\to\infty} X_n = \tfrac{2}{3}(X_1 - X_0) + X_0$

6.1.1 (b) $[a_n, b_n]$ is a nested set of intervals so both a_n and b_n are monotonic and bounded below by a_0 and above by b_0. Then since given $\varepsilon>0$ $\exists N$ such that $|a_N - b_N| < \varepsilon$ and $|a_N - b_N| = |a_N - L| + |L - b_N|$ for $L = \lim_{n\to\infty} a_n$, we have $L \underset{\varepsilon}{\approx} b$ so $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$. Then $\lim_{n\to\infty} \{a_0, b_0, a_1, b_1, \ldots\}$ exists and equals L.

6.2.2 Suppose for contradiction that $\{X_n\}$ has no cluster points. Then $X_n = a_i$ for a finite number of values of n. Similarly $X_n \in \{a_1, \ldots, a_K\}$ for a finite number of n. But $\{X_n\}$ is infinite so this is a contradiction.

6.3.1 (a) $b_n = \cos^2 a_n$ is bounded by $[0,1]$ for any sequence a_n, thus by Bolzano-Weierstrass b_n has a convergent subsequence.

(b) $b_n = \frac{a_n}{1+a_n}$ Solving $\frac{a_n}{1+a_n} = n$ gives $a_n = \frac{n}{1-n}$, so for this choice of a_n, b_n is un-bounded and contains no convergent sub-sequence.

(c) $b_n = \frac{1}{1+|a_n|}$ is bounded by $[0,1]$ so as in (a) b_n contains a convergent subsequence, for all a_n.

7.1.1 (a) $\sum_{2}^{\infty} \frac{1}{n^2-1} = \frac{1}{2}\sum_{2} \frac{1}{n-1} + \frac{1}{n+1}$

$= \frac{1}{2}\left[1 + \frac{1}{2} + \sum_{3}^{\infty} \frac{1}{k} - \frac{1}{k}\right]$

$= \frac{3}{4}$

(b) $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{n(n+2)} = \frac{1}{2} \sum_{1}^{\infty}\left(\frac{(-1)^{n-1}}{n} + \frac{(-1)^n}{n+2}\right)$

$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right) + \sum_{3}^{\infty} (-1)^k\left(\frac{1}{k} - \frac{1}{k}\right)\right]$

$= \frac{1}{4}$

(c) $\sum_{1}^{\infty} \frac{1}{n(n+k)} = \frac{1}{k} \sum_{1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+k}\right) = \frac{1}{k}\left[1 + \frac{1}{2} + \cdots + \frac{1}{k} + \sum_{j=k+1}^{\infty}\left(\frac{1}{j} - \frac{1}{j}\right)\right]$

$= \frac{1}{k}\left[1 + \frac{1}{2} + \cdots + \frac{1}{k}\right]$

7.2.2 $\sum a_n$ converges implies $a_n \to 0$ so in particular $a_n < 1$ for $n >> 1$. Then, since $a_n > 0$, $a_n^2 < a_n$ for such n. For some such n we have $\sum_{n}^{\infty} a_k^2$ converges by comparison to $\sum_{n}^{\infty} a_k$. Then by the tail convergence theorem $\sum a_k^2$ converges.

7.2.4 $(a_n - b_n)^2 \geq 0$

So $a_n^2 + b_n^2 \geq 2a_n b_n (> 0)$

By linearity the sum over the left hand side converges, hence by comparison the right hand side will converge as a series.