Math 311
HW 7 Solutions
April 11, 2008

7.3.2 Let $|b_n| < B$ for all n. By hypothesis, such a B exists. Then,

$$|a_n b_n| < |a_n| B \quad \text{for all } n,$$

and so $\sum |a_n b_n|$ converges by comparison with $\sum |a_n| B = B \sum |a_n|$ (linearity theorem). Then by absolute convergence theorem, $\sum a_n b_n$ converges.

7.3.3 a) $\sum |a_n|$ converges by hypothesis. Then let $S_N = \sum_{i=1}^{N} |a_{n_i}|$. We have $S_N \leq \sum_{i=1}^{n_N} |a_i|$ (since S_N is some but not necessrily all of the positive terms on the right). And $\sum_{i=1}^{n_N} \leq \sum_{i=1}^{\infty} |a_i|$, so S_N is bounded. Clearly S_N is increasing, hence by completeness property $\sum_i |a_{n_i}|$ converges. By absolute convergence theorem, $\sum_i a_{n_i}$ must converge.

7.3.3 b) Let $a_n = (-1)^n \frac{1}{n}$. Then $a_{2n} = \frac{1}{2n}$ and $\sum^{\infty} a_n = \ln 2$ is convergent but $\sum a_{2n} = \sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ is divergent.

7.4.1 b) $\sum_1 \frac{n^2}{2^n}$. Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{(n+1)^2 2^n}{n^2 2^{n+1}} \\
&= \lim_{n\to\infty} \frac{1}{2} \left(1 + \frac{2}{n} + \frac{1}{n^2} \right) \\
&= \frac{1}{2} < 1.
\end{aligned}$$

So the series converges.

7.4.1 d) $\sum_0 \frac{(n!)^2}{(2n)!}$ Ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{((n+1)!)^2 (2n)!}{(n!)^2 (2n+2)!} \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} \\
&= \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{4 + \frac{6}{n} + \frac{2}{n^2}} \\
&= \frac{1}{4} < 1.
\end{aligned}$$

So the series converges.

7.4.1 i) $\sum_0 \left(\frac{n}{n+2}\right)^{n^2}$. n^{th} root test:

$$\begin{aligned}\lim_{n\to\infty} |a_n|^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\frac{n}{n+2}\right)^n \\ &= \lim_{n\to\infty} \left(1+\frac{-2}{n+2}\right)^{n+2} \left(1+\frac{-2}{n+2}\right)^{-2} \\ &= \mathrm{e}^{-2} < 1.\end{aligned}$$

So the series converges.

7.4.1 j) $\sum_2 1/(\ln n)^p$ Integral test (can be used since $\frac{1}{(\ln\ n)^p} \geq 0$ and is decreasing.) Then, $\sum_2 \frac{1}{n(\ln\ n)^p}$ converges iff $\int_2^\infty \frac{1}{x(\ln x)^p}\mathrm{d}x$ is finite. For $p \neq 1$,

$$\int_2^\infty \frac{1}{x(\ln x)^p}\mathrm{d}x = \left.\frac{(\ln x)^{1-p}}{1-p}\right|_2^\infty = \lim_{n\to\infty} \frac{(\ln x)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}.$$

Since $\lim_{x\to\infty}(\ln x) = \infty$, then for $p < 1$ the series diverges and for $p > 1$ the series converges. For $p = 1$

$$\int_2^\infty \frac{1}{x \ln x}\mathrm{d}x = \ln(\ln x)|_2^\infty$$

which clearly is not finite. Hence convergence exactly on $p \in (1, \infty)$.

7.4.2 If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then for some N, $n > N$ implies $\left|\frac{a_{n+1}}{a_n}\right| > 1$ (i.e., let $\epsilon = [\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| - 1]/2$.) Then $|a_{n+1}| > a_n$ for $n \gg 1$, so a_n does not converge to 0. Therefore $\sum a_n$ fails the n^{th}-tern test for convergence, hence diverges.

7-3 We have $\left|\frac{a_{n+1}}{a_n}\right| \leq r$, hence

$$\left|\frac{a_k}{a_0}\right| = \prod_{n=0}^{k-1} \left|\frac{a_{n+1}}{a_n}\right| \leq r^k$$

($\prod$ is iterated multiplication opeator, as $\sum$ is interated addition operator.) Hence $|a_n| \leq |a_n|r^k$. Then by comparison to geometric series $|a_n|\sum r^2$ we have $\sum |a_n|$ is convergent. This implies $\sum a_k$ is convergent by the absolute convergence theorem, in fact absolutely convergent, which is the conclusion of the ratio test as well. However, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ need not exist to apply this (7-3) result, so this is stronger.

Consider the sequence

$$a_n = \left(\frac{r}{2}\right)^n \left(\frac{2}{3}\right)^{\frac{1}{2}(1+(-1)^n)}.$$

Then the ratio we're interested in is either $\frac{r}{3}$ or $\frac{3}{4}r$, depending on the parity (even or odd) of n, so (7-3) applies even though the ratio test cannot (since the ratio has no limit).