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1. Let $a_n = \sqrt{n}$.

(5pts) (a) Prove that for every $\varepsilon > 0$ there is $N \geq 1$ such that $|a_{n+1} - a_n| < \varepsilon$ if $n \geq N$.

Solution: Let $\varepsilon > 0$. Then $|a_{n+1} - a_n| < \varepsilon$ is equivalent with $|\sqrt{n+1} - \sqrt{n}| < \varepsilon$. Simplifying, this inequality is equivalent with

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \varepsilon.$$

Note that the previous expression is positive so we do not need the absolute value bars. Since

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}$$

if we choose $N = (1/(2\varepsilon))^2$, it follows that $|a_{n+1} - a_n| < \varepsilon$ if $n \geq N$.

(5pts) (b) Is a_n a Cauchy sequence? Why?

Solution: a_n is not a Cauchy sequence. To prove this, suppose by contradiction that a_n is Cauchy. Then a_n must be convergent by Theorem 6.4. The sequence a_n is, however, divergent. Thus it can not be Cauchy.

(7pts) 2. (a) Show that if $\sum a_n$ converges absolutely, then so does $\sum a_n^2$. Is this true without the hypothesis of absolute convergence (prove or give a counterexample)?

> **Solution:** If the series $\sum a_n$ converges absolutely it converges. Then $\lim_{n\to\infty} a_n = 0$. Then there is $N \geq 1$ such that $|a_n| < 1$ for all $n \geq N$. It follows that $a_n^2 \leq |a_n|$ for all $n \geq N$. The comparison test for positive series implies that $\sum_{n=N}^{\infty} a_n^2$ converges. From the tail theorem we conclude that $\sum a_n^2$ converges.
>
> The conclusion fails without the hypothesis of absolute convergence. Consider $\sum \frac{(-1)^n}{\sqrt{n}}$. This series is convergent by the Cauchy test. The series $\sum \frac{1}{n}$, however, diverges (it is a p-series with $p = 1$).

(3pts) (b) If $\sum a_n$ converges and $a_n \geq 0$, does it follow that $\sum \sqrt{a_n}$ converges? Prove or give a counterexample.

> **Solution:** If $\sum a_n$ converges and $a_n \geq 0$, it does not follow that $\sum \sqrt{a_n}$ converges. For example, $\sum \frac{1}{n^2}$ converges (p-series with $p = 2$) but $\sum \frac{1}{n}$ diverges.

(20pts) 3. Find the radius of convergence of $\sum_{n=0}^{\infty}(\sin n)x^n$ (with proof).

Solution: The main point about this problem is that we can not use the root or the ratio test to determine the radius of convergence of this series (because the sequences $(\sin(n))^{(1/n)}$ and $\frac{\sin(n+1)}{\sin(n)}$ are divergent).

Notice that $|\sin(n)x^n| \leq x^n$ for all $n \geq 1$. If $|x| < 1$ we know that $\sum x^n$ converges absolutely. The comparison theorem implies that $\sum \sin(n)x^n$ converges absolutely. For $x = 1$, $\sum \sin(n)$ diverges since $\{\sin(n)\}$ does not converge to 0. Thus $R =$ by Theorem-Definition 8.1.

(10pts) 4. Determine if the series

$$\sum_{n=1}^{\infty} \ln \frac{n}{n+2}$$

is convergent. If yes, find it's value.

Solution: Since $\ln \frac{n}{n+2} = \ln(n) - \ln(n+2)$ we see that the series is a telescoping series. We compute the nth partial sum as follows:

$$s_n = \sum_{k=1}^{n} \big(\ln(k) - \ln(k+2)\big) = \ln(1) + \ln(2) - \ln(n+1) - \ln(n+2).$$

Then

$$\lim_{n\to\infty} s_n = \ln(2) - \lim_{n\to\infty} \big(\ln(n+1) + \ln(n+2)\big) = -\infty.$$

Thus the series is divergent.

5. Let b_n be a decreasing sequence with $\lim_{n\to\infty} b_n = 0$.

(5pts) (a) Prove that $\sum_{n=1}^{\infty}(b_n - b_{n+1})$ converges.

Solution: We notice this is a telescoping series. The partial sum equals

$$s_n = (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_n - b_{n+1}) = b_1 - b_{n+1}.$$

Since $\lim_{n\to\infty} b_n = 0$ it follows that the sequence of partial sums is convergent and equals b_1. Thus $\sum(b_n - b_{n+1})$ converges.

(10pts) (b) Let a_n be a bounded sequence. Prove that $\sum_{n=1}^{\infty} a_n(b_n - b_{n+1})$ converges.

Solution: By hypothesis, there exists $M > 0$ such that $|a_n| \leq M$ for all $n \geq 1$. Since $b_n - b_{n+1} \geq 0$ we have that $|a_n(b_n - b_{n+1})| \leq M(b_n - b_{n+1})$. By the linearity theorem $\sum M(b_n - b_{n+1})$ converges. By the comparison theorem, $\sum |a_n(b_n - b_{n+1})|$ converges. Thus $\sum_{n=1}^{\infty} a_n(b_n - b_{n+1})$ converges.

6. Prove the following two statements:

(10pts) (a) Every real number is a cluster point of some sequence of rational numbers.

> **Solution:** Let r be a real number. For any $n \geq 1$, Theorem 2.5 on page 25 implies that there is a *rational number* a_n such that $r < a_n < r + \frac{1}{n}$. By the squeeze theorem, $\lim_{n\to\infty} a_n = r$. Theorem 6.2 implies that r is a cluster point for the sequence a_n.

(5pts) (b) Every real number is a cluster point of some sequence of irrational numbers.

> **Solution:** The proof is similar with the previous part. If r is any real number, Theorem 2.5 implies that there is an *irrational number* a_n with $r < a_n < r + \frac{1}{n}$ for all $n \geq 1$. Thus $r = \lim_{n\to\infty} a_n$ and r is a cluster point of a sequence of irrational numbers.

7. Give examples for the following or explain why no example exists.

(3pts) (a) A series that has bounded partial sums but does not converge.

> **Solution:** Consider the sequence $a_n = (-1)^n$ for all $n \geq 1$. Then $s_{2k+1} = -1$ and $s_{2k} = 0$ for all $k \geq 1$. Thus $\{ s_n \}$ is a bounded sequence which diverges.

(4pts) (b) A sequence which has an infinite number of cluster points.

> **Solution:** Example 6.2A a) in the textbook says that for the sequence $1; 1, 2; 1, 2, 3; \ldots$ every integer is a cluster point.

(3pts) (c) A power series whose radius of convergence is 2.

> **Solution:** Consider the power series $\sum \frac{x^n}{2^n}$. We compute the radius of convergence using the ratio test:
> $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{2}.$$
> Thus the series is absolutely convergent if $|x| < 2$ and divergent for $|x| > 2$. Thus $R = 2$.

(10pts) 8. Prove, using the definition, that if $\{ a_n \}$ and $\{ b_n \}$ are Cauchy sequences then $\{ a_n b_n \}$ is Cauchy.

Solution: We proved in class that a Cauchy sequence is bounded (see also part (A) of the proof of Theorem 6.4). Thus there are $M_1 > 0$ and $M_2 > 0$ such that $|a_n| \leq M_1$ and $|b_n| \leq M_2$ for all $n \geq 1$. Let $M = \max\{M_1, M_2\}$. Thus $|a_n| \leq M$ and $|b_n| \leq M$ for all $n \geq 1$. Let $\varepsilon > 0$. Since the sequence a_n is Cauchy there exists $N_1 \geq 1$ such that $|a_n - a_m| < \frac{\varepsilon}{2M}$ for all $m, n \geq N_1$. Similarly there exists $N_2 \geq 1$ such that $|b_n - b_m| < \frac{\varepsilon}{2M}$ for all $m, n \geq N_2$. Let $N = \max\{N_1, N_2\}$. If $n, m \geq N$ we have

$$\begin{aligned} |a_n b_n - a_m b_m| &= |a_n b_n - a_n b_m + a_n b_m - a_m b_m| \\ &\leq |a_n b_n - a_n b_m| + |a_n b_m - a_m b_m| \\ &= |a_n||b_n - b_m| + |b_m||a_n - a_m| \\ &< M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, $\{a_n b_n\}$ is a Cauchy sequence.

 End of exam.