Finding Homologies of Sphere Quotients

Marisa Belk, Cornell University

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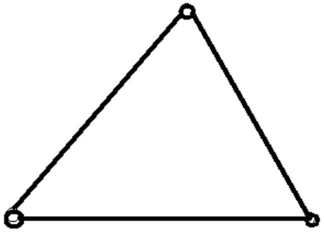
Ex: E := $\{v_{1,} \cdots, v_k\}$ are the columns of a matrix,
I := linearly independent subsets of E (over a given field)

Matroids from Graphs

Any graph yields a matroid: the edges of the graph are the elements (edges) of the matroid

Independent sets are subsets of spanning trees, so a set is independent if and only if it contains no circuits.

Examples:



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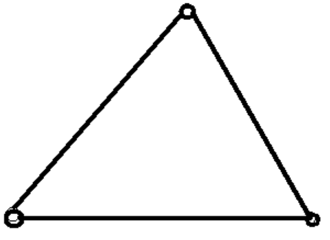

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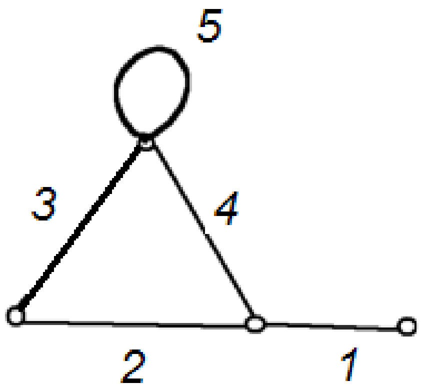
-Any subset of the edges of a tree is independent

-Every graphical matroid can be represented by a the columns of a matrix (but the converse is not true)

Terminology

Here are two representations of the same matroid:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$



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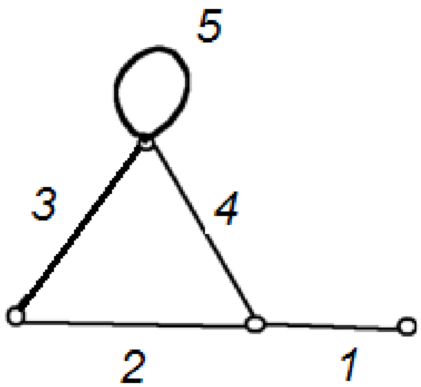

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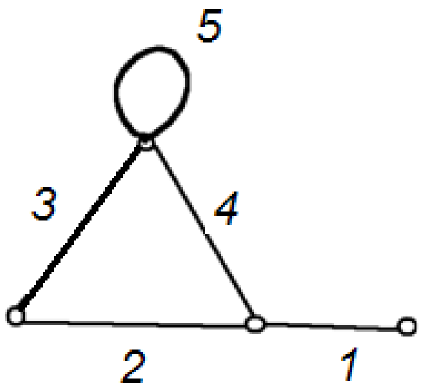

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The middle three vectors form a **circuit**, or minimal dependent set

The fifth element is called a **loop**: it forms a circuit by itself.

The first vector, on the other hand, is not contained in any circuits. Such an element is called a **coloop** (or **isthmus**).

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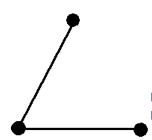
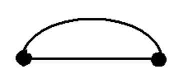
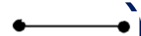
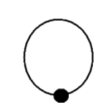
T( x, y) = T ( x, y) + T( ; x, y)

= x*T( ; x, y) + T(;  x, y) + T(; x,  y)

= x^2 + x + y

Binary Spherical Quotients

Orthogonal matrices represent symmetries of the sphere. In the case where the orthogonal matrices form a group isomorphic to $(Z_2)^r$, we call the quotient space a BSQ.

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Ex: The following elements of O(3) act on S^2, yielding BSQ's

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Reflection over the z-axis,
the resulting quotient is a hemisphere

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Rotation by 180°,
the resulting football-shaped orbifold

RP^n is a BSQ It is the quotient of S^{n-1} by the antipodal map $x \rightarrow -x$, which is a diagonal matrix of -1's.

What can we say about Homology?

Since we can give the sphere a very nice simplicial structure, and we are acting by a finite group, we have the following theorem:

Theorem: If G is a finite group, and F be a field such that char(F) = 0 or char(F) $\nmid |G|$.

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Since the sphere only has homology in dimension n, the rational homology is zero everywhere but n.
$H_n(X) = 0 \Leftrightarrow$ any of the involutions are orientation reversing

Conclusion: Only homology over Z_2 or Z will be of interest.

The Matroid Corresponding to a BSQ

If $A \in O(3)$ represents an element of $(Z_2)^r$, then $A^2 = I$, thus A is conjugate to a diagonal matrix of ± 1's.

We can get a $r \times n$ binary matrix M from the action by using these diagonals as rows. We convert 1's to 0's and -1's to 1's

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

he matroid [0 0 1] (coloop & 2 loops)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The matroid [0 1 1] (loop & 2-circuit)

RP^n The matroid [1 1 1] $(U_{1,3})$

What the Matroid Gives Us

Theorem: There is a one-to-one correspondence between binary matroids (up to isomorphism) and binary spherical quotients (up to isometry)

Fact: If X is a BSQ, and the corresponding M contains a coloop, the quotient is contractible; $H_i(X, Z_2) = 0$

Fact: If X is a BSQ, and the corresponding M contains a loop e, then X is the suspension of Y, corresponding to M-e ;
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Theorem[Swartz]: Let X be BSQ with corresponding matroid M. Let $P_X(t) = \sum_m \dim_{Z_2}(H_m(X, Z_2))t^m$ be the Poincare polynomial. Then $P_M(t) = t^{r-1}\, T(M; 0, t)$.

Proof: Meyer-Vietoris Sequence + a lot of work

Odd Primes

Take $(Z_p)^r$ y S^{2n-1} an effective/faithful action.

This gives us a representation $(Z_p)^r \to O(2n)$

Though not necessarily diagonalizable over R, we can represent each element of $(Z_p)^r$ as a block diagonal matrix:
$A = \mathrm{diag}(A_1, \cdots, A_n)$ where each A_i is a 2x2 rotation matrix.

Thus A is rotating each circle of the join $S^{2n-1} = S^1 * \cdots * S^1$ at some speed k_j $1 \le j \le n$ as A_j is rotation by $e^{k_j(2\pi/p)}$

We once again use this "diagonal", i.e. the values of k_j for a given A, to create a row a matrix over Z_p, and thus a matroid.

Although this correspondence is NOT one-to-one, we can use it to find the homology over Z_p.

Quotients By Tori

Once again, representation theory gives us an answer: an S^1 – generator of $T^r = S^1 \times \cdots \times S^1$ act on the circles of $S^{2n-1} = S^1 * \cdots * S^1$ by rotation with wrapping number k_j

We get an integer $r \times n$ matrix by creating a row out of the the k_j's for each generator of T^r.

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If $$T(M;0,t) = \sum_{i=1}^{n(M)} a_i t^i \qquad \widetilde{P}_X(t) = t^r\left(\sum_{i=1}^{n(M)} a_i t^{2i-1}\right)$$

Notice that the homology only appears in every other degree.
I haven't tried integer homology yet!

Quotients by Cyclic Groups

Rational homology is again easy to compute (since we are acting by a finite group)

Integer homology is difficult to compute, as we would need more pieces: the homology over all the primes that divide the order of the group

I have written a program that computes the homologies of given examples (of sufficiently small dimension).

I have a conjecture….but it isn’t easy to phrase in terms of matroids. Stay tuned for next time!