Matroids, Invariants, and Colorings

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- Matroids are a good generalization since they share a lot of the important properties of these structures, including the *Tutte Polynomial*
- Matroids also come from a lot of other combinatorial structures like really nice lattices and simplicial complexes

A Linear Independence Refresher

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- We can tell if a set of n vectors is linearly independent by putting them in the columns of a matrix and row reducing. The matrix has rank n if and only if the vectors are independent.

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  - 3) If $A, B \in \mathcal{I}$, and $|A| > |B|$, then $\exists x \in A$ such that $B \cup \{x\} \in \mathcal{I}$
- For any $A \subseteq E$, define the rank of A, denoted $r(A)$ to be the size of the biggest independent set in A
- If $e \in E$ such that $r(\{e\}) = 0$, then e is called a *loop*
- If $e \in E$ such that for all $A \subseteq E$, $r(A \cup e) = r(A) + 1$ then e is called a coloop

Getting a Matroid from a Matrix

Start with a matrix $M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

Take $E :=$ the column vectors of this matrix

Take $\mathcal{I} :=$ linearly independent subsets of the columns

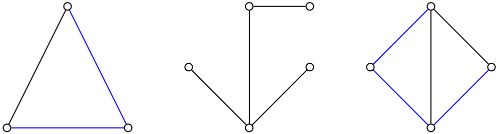
Then $rank(E) = rank(M)$

- Which element is a loop?
- Which element is a coloop?
- Which elements for a *circuit*, or minimal dependent set?

A Graph Theory Refresher/Lesson

A Graph G consists of a vertex set V and edge set $E \subseteq V \times V$

- A *path* in a graph is a sequence of connected edges
- A *cycle* is a path that starts and ends at the same point
- A *tree* is a graph with no cycles.
- A *spanning tree*, this is a subgraph with no cycles that hits every vertex. If G has n vertices, every spanning tree has $n - 1$ edges.



Getting a Matroid from a Graph

Start with a graph G. The edges E of the matroid associated to the graph are the edges of the graph.

A set of edges is independent if and only if it contains no circuits (it is a subset of a spanning tree)

If $|V| = n$, then $r(E) = n - 1$

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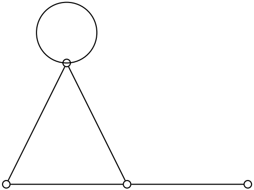
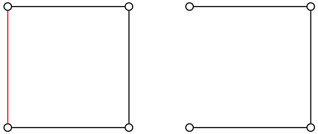


Deletion and Contraction

Deletion:

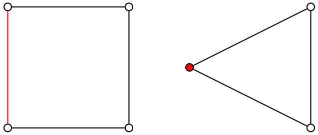
- Remove an edge from a graph, leaving the rest of the graph the same
- Deleting a column from a matrix, resulting in a smaller matrix
- Creating a new matroid $M - e$ with edge set $E - e$ and $A \in \mathcal{I}_{M-e} \Leftrightarrow A \in \mathcal{I}$



Contraction

Contraction:

- In a graph, contract the edge to a vertex as below
- For a matroid, the edge set of M/e is again $E - e$, but the independence relation has changed:
  A set $A \subseteq E_{M/e}$ is in $\mathcal{I}_{M/e} \Leftrightarrow A \cup e \in \mathcal{I}$



From the matroid perspective, deletion and contraction are the same for loops and coloops

The Tutte Polynomial

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The Tutte Polynomial is defined recursively as the unique 2-variable polynomial such that:

- $T(\text{a single loop}) = y$
- $T(\text{a single coloop}) = x$
- When e is a loop or coloop, $T(M) = T(e) * T(M - e)$
- For e neither, $T(M) = T(M - e) + T(M/e)$

Some Information Hidden in the Tutte Polynomial

Some nice properties of the tutte poly:
If we have the Tutte Polynomial $T(M; x, y)$ we can find....

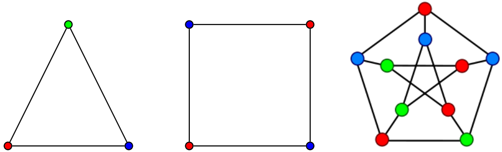
- $r(E)$ is the highest degree of x in the Tutte polynomial
- The nullity of M, $|E| - r(E)$, is the highest degree of y in the Tutte polynomial
- $T(M; 1, 1)$ the total number of bases or spanning trees that exist in M
- $T(M; 2, 1) = |\mathcal{I}|$, the total number of independent sets in M (subtrees)
- $T(M; 2, 2) = 2^{|E|}$, number of subsets of M

Graph Colorings

A graph coloring is a selection of colors for the vertices so that every edge touches two different colors.

A graph is called *m-colorable* if you can color it with *m* colors obeying this rule.

For example, every tree is 2-colorable (just alternate the colors). The square is 2-colorable, but you need 3 colors for a triangle (and the Peterson Graph):



How Many Colors do we Need?

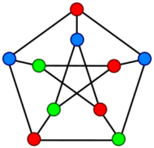
It's a common problem in graph theory to ask what the minimum number of colors is for a given graph. Also, if you can color a graph with m colors, how many different ways are there to do it?

This information is expressed by the *chromatic polynomial*. $\chi_G(\lambda) =$ the number of ways to colorg G with λ colors.

Let's do the triangle:

- $\chi_G(1) = 0$
- $\chi_G(2) = 0$
- $\chi_G(3) = 6$
- $\chi_G(\lambda) = \lambda(\lambda - 1)(\lambda - 2)$

But this gets hard quickly...



The Chromatic polynomial of the Peterson graph is
$t(t-1)(t-2)(t^7 - 12t^6 + 67t^5 - 230t^4 + 529t^3 - 814t^2 + 775t - 352)$

Yikes!

Saved by Tutte

In fact, $\chi_G(\lambda) = \lambda^{k(G)}(-1)^{|V(G)|-k(G)}T(M; 1-\lambda, 0)$
Where $k(G)$ is the number of connected compodents,
$V(G)$ is the vertex set of G

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  The colorings of G are precisely colorings of $G - e$ where the endpoints of e, u and v, are colored differently. So, if we choose one of λ colors for u, there are $\lambda - 1$ colors possible for v (instead of λ). Thus, $\chi_G = \frac{\lambda-1}{\lambda}\chi_{G-e}$, so $\chi_G = \frac{(1-\lambda)}{-\lambda}\chi_{G-e}$

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- If e was a loop, you can remove it and multiply the chromatic number by zero
- If e is a coloop of G, its deletion disconnects the graph, and $\chi_G = \frac{(1-\lambda)}{-\lambda}\chi_{G-e}$
- If e is not a loop or coloop, $G - e$ has the same connectivity as G. Partition the colorings of $G - e$ into those where u, v are colored the same, $\chi_{G/e}(\lambda)$, and u and v are colored differently $\chi_G(\lambda)$ Thus, $\chi_G(\lambda) = -\chi_{G/e}(\lambda) + \chi_{G-e}(\lambda)$

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Then, the number of regions $= T(M; 2, 0)$

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Answer: $(1-p)^{|E|-r(M)} p^{r(M)} T(M; 1, \frac{1}{1-p})$

Other Applications

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$$\widetilde{P}_X(t) = \sum_{m=0}^{n-1} \dim_{Z_2} \tilde{H}_m(X, Z_2) t^m = t^{r-1} T(M; 0, t)$$

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If X is a quotient of a sphere by a torus,

$$\widetilde{P}_X(t) = t^r (\sum_{i=1}^{n(M)} a_i t^{2i-1}), \text{ where } T(M; 0, t) = \sum_{i=1}^{n(M)} a_i t^i$$

The End

T. Brylawski and J. G. Oxley. The Tutte polynomial and its Applications. In N. L. White, editor, Matroid Applications, pages 123 225. Cambridge University Press, 1992

J. G. Oxley. Matroid Theory. Oxford University Press, Oxford, 1992

Ed Swartz. Matroids and quotients of spheres, Math. Zeit., 241 (2002), 247-269.