Weekend Activity: Sigma Notation and Series

Goal: This activity reviews Sigma notation, which will be used constantly from next week forward. We review the rules and definitions of sigma notation, and introduce the definition of a series. **There will be a quiz about sequences and series on Tuesday!**

Remember defining integrals? First, we broke our interval up into subintervals. We multiplied the width of each subinterval by a height of the function in that subinterval, and added them all up to get a rectangle approximation of the area. The Riemann sum: $S_n = \sum_{k=1}^{n} f(c_k)\Delta x_k$ (see page 314) was our approximation using n rectangles.

Then $\lim_{n\to\infty} S_n = \int_a^b f(x)dx$

Of course, finding these sums and their limits was a difficult task. We only had a few tools at our disposal: the algebra rules of finite sums (p 308) and a few selected sums (p. 309). We have more recently learned that these sums can be taken over infinite intervals or up to vertical asymptotes, where the question of their convergence is an interesting one.

1) Briefly look through section 5.2 as a reminder, being sure to review these rules.

2) Complete the following exercises on a separate sheet, then check your answers using the back of the book or a solution manual. Section 5.2 : $1, 3, 7, 15, 17, 31$.

The sums you computed above were all *finite sums*. A *series* is a sum of infinitely many numbers. It is usually written by $\sum_{n=1}^{\infty} a_n$. Of course, if we want to add up infinitely many numbers, we need a list of infinitely many numbers. Thus the a_n above is a *sequence.*

So a series is defined by the sequence of numbers that it adds up. We can also use the series itself to define another special sequence: the **sequence of partial sums**.

$s_{\mathbf{n}} = \sum_{k=1}^{\mathbf{n}} a_k$ is the n^{th} term of the sequence of partial sums

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

3) Find s_1, s_2, s_3, s_4, and s_5; Do not simplify your answers.

4) Note that s_n an increasing sequence (each term is bigger than the previous term). Why should we expect this to occur? Will this be true of the partial sums $s_n = \sum_{k=1}^{n} a_k$ for ANY sequence a_n?

So although the terms of $a_n = \frac{1}{n^2}$ get smaller and smaller, the terms of s_n keep getting get bigger and bigger. It's clear that $\lim_{n\to\infty} \dfrac{1}{n^2} = 0$, but what is $\lim_{n\to\infty} s_n$? Unfortunately, iit could be just about anything!

5) Explain why the limit of s_n is so unpredictable, even though the sequence is increasing.

6) Using your calculator, approximate $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ by finding s_{10} as a decimal.

7) The actual value of this sum is $\pi^2/6$. Estimate $\pi^2/6$ using your calculator, and compare the results to your approximation. How do the two compare? How would you know which is bigger without even checking the numbers?

We say **a series converges if and only if its sequence of partial sums converges.** Another way to say this would be $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} s_k$

We will be devoting a lot of time to the question of whether or not a given series **converges**. The answer to this question can be tricky, but we will learn a number of techniques that can help us answer it. It will not always be clear which technique to use, so we will have to practice. In this regard, learning the tests for convergence of a series will be a lot like learning the techniques of integration. The problems below will give you a taste of the different ways to determine if a sequence converges. Once you've finished a problem, think about how your methods would be generalized.

8) Use the definition of convergence to show that the series $\sum_{k=1}^{\infty} 0$ converges to zero. Begin by finding a formula for the sequence of partial sums s_n.

9) Use the definition of convergence to show that the series $\sum_{k=1}^{\infty} c$ diverges for any constant $c \neq 0$. Begin by finding a formula for the sequence of partial sums s_n.

10) If we know that the series $\sum_{n=1}^{\infty} a_n$ converges, what (if anything) can we say about

a) $\lim_{n\to\infty} s_n$?

b) $\lim_{n\to\infty} a_n$?

If $\sum_{k=1}^{\infty} a_n = A$ and $\sum_{k=1}^{\infty} b_n = B$ (both converge to constants), what will $\sum_{k=1}^{\infty} (2a_n + b_n)$ do?

11) I have told you that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Use this fact to determine whether $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges or diverges. Justify your answer.

12) Use the fact that $\int_1^\infty x^{-3} dx$ converges to argue that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.
Hint: Draw a picture of $\int_1^\infty x^{-3} dx$ being underestimated by rectangles of width 1.

13) Let $a_n = (-1)^{n+1} \frac{1}{n^3}$. Consider $\sum_{n=1}^{\infty} a_n$. Note that $s_5 = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \frac{1}{125}$.
$\sum_{n=1}^{\infty} a_n$ is called an *alternating series* since the terms of the sum *alternate* between being positive and negative. Does $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$ converge or diverge? Justify your asnwer.

14) Based on your knowledge of improper integrals and your experience in (12), can you think of a series that probably diverges even though the terms go to zero?

Weekend Activity: To Converge or Not to Converge...

Complete this activity by Nov 7. A **convergence test** is a method for determining whether a series converges or diverges. Each method can be useful on different types of series. Some of the tests are not always practical, or can give inconclusive results. They generally reduce the problem to another that you may be able to solve: the convergence of a limit, another simpler series, or an improper integral.

A) Attach this to last weekend's activity. Also attach some extra paper for C-E.
B) Use your experience from the last activity to fill in the blanks below. You will produce your own versions of four convergence tests.
C) For each test, come up with a series not already in this packet that the test proves is convergent or divergent
D) Come up with a series or situation for which the test is not useful/inconclusive.
E) Match each test to the best name. Given in no particular order, the names of these four tests are: the integral test, the absolute convergence test, the comparison test, and the nth term test.

Note: When it comes to convergence, only the long term behavior of the series matters. To make the tests a little more general, we restrict our attention to the terms beyond a certain integer N.

1) Make the most general test that you can, based on your answer to 10b:

If $\lim_{n\to\infty} a_n$ ______________, then $\sum_{n=1}^{\infty} a_n$ ______________.

2) Create a test based on the technique you used in question 11: Let a_n and b_n be two sequences and N a natural number be such that ______________ for all $n \geq N$. Then if ______________ converges, then ______________ converges; if ______________ diverges, then ______________diverges.

3) Create a test based on the technique you used in question 12: Let $f(x)$ be a continuous, positive, decreasing function for all $x \geq N$. Let a_n be a sequence such that ______________ for all $n \geq N$. Then if ______________ converges, ______________ converges. Additionally, if ______________ diverges, then ______________ diverges. (you can use overestimating rectangles to prove this second part)

4) Create a test based on your solution to question 13: Let $\sum_{i=1}^{\infty} a_n$ be an alternating series. Then if ______________ converges, $\sum_{i=1}^{\infty} a_n$ converges.