Homework 1: Math 6710 Fall 2010

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August 30, 2010

Due in class on Friday, September 3.

Update: To be consistent with the text, you should assume that all measure spaces are σ-finite.

Throughout this homework $(\Omega, \mathcal{F}, \mu)$ is an arbitrary σ**-finite** measure space. measure space. All functions $f : \Omega \to \mathbb{R}$ or $f : \Omega \to [0, \infty]$ are assumed to be Borel measurable, i.e. $f^{-1}(B) \in \mathcal{F}$ for all Borel sets $B \subset \mathbb{R}$ or $B \subset [0, \infty]$.

1. (Durrett 1.4.1). Prove: If $f : \Omega \to [0, \infty]$ is a nonnegative measurable function with $\int f\, d\mu = 0$, then $f = 0$ a.e.

2. Suppose $f : \Omega \to [0, \infty]$ is a nonnegative measurable function. For any $A \in \mathcal{F}$, define $\nu(A) = \int_A f\, d\mu$. Prove:

   (a) (Durrett 1.5.4) ν is a countably additive measure on $(\Omega, \mathcal{F})$.

   (b) (Like Durrett 1.6.8) If $g : \Omega \to [0, \infty]$ is a nonnegative measurable function, then $\int g\, d\nu = \int fg\, d\mu$.

3. (Like Durrett 1.5.8) Let m be Lebesgue measure on $\mathbb{R}$, and suppose $f \in L^1(\mathbb{R}, m)$. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \int_{(-\infty,x)} f\, dm$. Show that F is continuous. (Hint: Dominated convergence. Remark: F actually has the stronger property of being *absolutely continuous*.)

4. (Durrett 1.5.6) Prove: If $g_m : \Omega \to [0, \infty]$, $m = 1, 2, \dots$ is a sequence of nonnegative measurable functions, then

$$\int \sum_{m=1}^{\infty} g_m\, d\mu = \sum_{m=1}^{\infty} \int g_m\, d\mu.$$

   That is, we may interchange sums and integrals of nonnegative functions. Please do not use Tonelli's theorem (of which this is a special case).

5. (Durrett 1.5.10) Prove: If $g_m : \Omega \to \mathbb{R}$ are measurable functions and $\sum_{m=1}^{\infty} \int |g_m|\, d\mu < \infty$, then

$$\int \sum_{m=1}^{\infty} g_m\, d\mu = \sum_{m=1}^{\infty} \int g_m\, d\mu.$$

   (Please do not use Fubini's theorem, of which this is a special case.) Give an explicit example to show that this may be false if $\sum_{m=1}^{\infty} \int |g_m|\, d\mu = \infty$.