Homework 2: Math 6710 Fall 2010

Due in class on Friday, September 10.

1. It is very easy to check that an arbitrary intersection of σ-fields is again a σ-field. In this problem, you will show that the same does not hold for unions.

   (a) Give an example of a set Ω and σ-fields $\mathcal{F}_1, \mathcal{F}_2 \subset 2^\Omega$ such that $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-field.

   (b) Give an example of a set Ω and a nested sequence of σ-fields $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is not a σ-field.

2. (Durrett 1.6.8) Suppose $f : \mathbb{R} \to [0, \infty)$ is a probability density function (i.e. $\int_\mathbb{R} f\,dm = 1$, where m is Lebesgue measure), and let $\mu(B) := \int_B f\,dm$ be the corresponding probability measure. Show that for any measurable $g : \mathbb{R} \to \mathbb{R}$ with $g \ge 0$ or $\int_\mathbb{R} |g|\,d\mu < \infty$, we have $\int_\mathbb{R} g\,d\mu = \int_\mathbb{R} fg\,dm$. (See the proof of Theorem 1.6.9 for the "standard mantra." This is sometimes used as the definition of expectation of a continuous random variable in undergraduate texts; this exercise shows the definition is consistent.)

3. (Durrett 1.6.1) Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is *strictly* convex, i.e.

$$\varphi(tx + (1-t)y) < t\varphi(x) + (1-t)\varphi(y)$$

   for all $x \ne y$ and $0 < t < 1$. ("Convex" only requires $\le$ in the above inequality.) Show that under this assumption, equality holds in Jensen's inequality only in the trivial case that X is a.s. constant. That is, if X and $\varphi(X)$ are integrable and $E[\varphi(X)] = \varphi(EX)$ then $X = EX$ a.s.

4. (Durrett 1.2.3) Show that a distribution function has at most countably many discontinuities. That is, for any random variable there are at most countably many real numbers x with $P(X = x) > 0$.

5. (Durrett 1.2.4) Show that if $F(x) = P(X \le x)$ is continuous then $Y = F(X)$ has a uniform distribution on $(0, 1)$; i.e. $P(Y \le y) = y$ for $y \in [0, 1]$. Give a counterexample to show that this need not be the case if F is not continuous. (Remark: the "inverse" of this statement is also true: if U has a uniform distribution on $(0, 1)$ and F is a distribution function, then $F^{-1}(U)$ is a random variable with distribution function F. You have to define F^{-1} appropriately since F need not be 1-1 in general. This is useful in programming if you know how to generate uniform random variables but need some other distribution.)