Homework 3: Math 6710 Fall 2010

Due in class on Friday, September 17.

1. Let X be a random variable. Prove that the following are equivalent:

   (a) X is independent of every random variable Y.

   (b) X is independent of itself.

   (c) For all events $A \in \sigma(X)$, $P(A) = 0$ or $P(A) = 1$. (We say the σ-field $\sigma(X)$ is **almost trivial**.)

   (d) There exists a constant $c \in \mathbb{R}$ such that $X = c$ a.s.

   (Remark: We proved in class that the "most independent" events are those which have probability 0 or 1. This problem gives the corresponding statement for random variables. We will see more of the notion of an almost trivial σ-field when we discuss the Kolmogorov 0-1 law, and later, the Blumenthal 0-1 law.)

2. (Like Durrett 2.1.4) Let $g_1, \dots, g_n$ be probability density functions (i.e. $g_i : \mathbb{R} \to [0, \infty)$ is measurable and $\int_{\mathbb{R}} g_i \, dm = 1$), and define $f : \mathbb{R}^n \to [0, \infty)$ by $f(x_1, \dots, x_n) = g_1(x_1) \dots g_n(x_n)$. Show that $\mathbf{X} = (X_1, \dots, X_n)$ is a random vector with density f if and only if $X_1, \dots, X_n$ are independent random variables where X_i has density g_i.

   (Remark: In Durrett's version, the g_i are not assumed to be probability densities, i.e. it is not assumed that $\int_{\mathbb{R}} g_i \, dm = 1$. However, one can just rescale them to achieve this.)

3. (a) Let $\mathcal{G}$ be a σ-field, and let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \dots$ be an increasing sequence of σ-fields. Suppose that for each n, $\mathcal{G}$ and $\mathcal{G}_n$ are independent. Let $\mathcal{G}_\infty = \sigma(\mathcal{G}_1, \mathcal{G}_2, \dots)$ be the σ-field generated by the $\mathcal{G}_n$ (i.e. the smallest σ-field such that $\mathcal{G}_n \subset \mathcal{G}_\infty$ for all n). Show that $\mathcal{G}$ and $\mathcal{G}_\infty$ are independent. (Hint: $\bigcup_{n=1}^\infty \mathcal{G}_n$ is a π-system.)

   (b) Let Y be a random variable, and $X_1, X_2, \dots$ a sequence of random variables such that Y is independent of $(X_1, \dots, X_n)$ for each n. Show that Y is independent of $\sup_n X_n$. (It is also independent of $\inf_n X_n$, $\limsup_n X_n$, etc.) (Hint: Use part (a).)

4. (Durrett 2.1.13) Show that if X, Y are independent discrete random variables, then

$$P(X + Y = n) = \sum_m P(X = m) P(Y = n - m)$$

5. (Durrett 2.1.14) Recall that a random variable Z has the Poisson distribution with parameter λ if $P(Z = k) = e^{-\lambda} \lambda^k / k!$ for $k = 0, 1, 2, \dots$; we write $Z \sim \text{Poisson}(\lambda)$. Suppose X, Y are independent with $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$. Use the previous exercise to show that $X + Y \sim \text{Poisson}(\lambda + \mu)$.