Homework 4: Math 6710 Fall 2010

Due in class on Friday, September 24.

1. (Durrett 2.3.12) Let A_n be a sequence of independent events with $P(A_n) < 1$ for all n. Show that $P(\bigcup A_n) = 1$ implies $P(A_n \text{i.o.}) = 1$.

2. Suppose A_n is any sequence of events with $P(A_n) \geq \epsilon > 0$ for all n. Show that $P(A_n \text{ i.o}) \geq \epsilon$. (This is sort of a partial converse of the first Borel-Cantelli lemma.)

3. Suppose $X_1, X_2, \ldots$ are iid.

   (a) The **essential supremum** of a random variable X is defined as

   $$\operatorname{esssup} X := \sup\{x \in \mathbb{R} : P(X > x) > 0\}.$$

   (This is almost like the supremum of X as a real-valued function except that it disregards events of probability zero.) Show that $\limsup_{n\to\infty} X_n = \operatorname{esssup} X_1$ a.s. So, if you want to know the essential supremum of a random variable, generate an iid sequence with that distribution and look at its limsup.

   (b) The **essential range** of a random variable is defined as the following set of real numbers:

   $$\operatorname{essran} X = \bigcap \{F \subset \mathbb{R} : F \text{ closed}, P(X \in F) = 1\}.$$

   (This is like the closure of the range of X as a real-valued function except that it disregards events of probability zero.) Show that, with probability one, the sequence X_n is a dense subset of $\operatorname{essran} X_1$. In other words, if

   $$A = \{\omega : \{X_1(\omega), X_2(\omega), \ldots\} \text{ is dense in } \operatorname{essran} X_1\},$$

   show that $P(A) = 1$. So, if you want to know the essential range of a random variable, generate an iid sequence with that distribution, collect up all the values you see, and take the closure.

   Hint: For any $x \in \operatorname{essran} X_1$ and positive integer m, let $A_{x,m}$ denote the event that $|X_n - x| < 1/m$ for some n. Show $P(A_{x,m}) = 1$. Then let $\{x_k\}$ be a countable dense subset of $\operatorname{essran} X_1$ and check that $A = \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} A_{x_k,m}$.

4. For this problem, suppose we are working on a *discrete* probability space. That is, Ω is a countable set (finite or countably infinite) and $\mathcal{F} = 2^{\Omega}$.

   (a) (Durrett 2.3.16) Suppose X_n is a sequence of random variables on a *discrete* probability space such that $X_n \to X$ in probability. Show that $X_n \to X$ almost surely. So, on a discrete probability space, convergence i.p. and a.s. are equivalent (the reverse implication holds in any probability space and was shown in class).

   (b) (Bonus problem) Let X_n be an *iid* sequence of random variables on a discrete probability space. Show that there is a constant $c \in \mathbb{R}$ such that $X_n = c$ a.s. So, discrete probability spaces are too small to support any really interesting models.