Homework 5: Math 6710 Fall 2010

Due in class on Friday, October 1. You may assume the strong law of large numbers for iid integrable random variables (e.g. Theorem 2.4.1) although we proved a slightly weaker version in class.

Recall $\mathrm{Var}(X) = E[(X - EX)^2] = E[X^2] - E[X]^2$. Note that $\mathrm{Var}(X) < \infty$ iff $X \in L^2$.

1. (Durrett 2.2.1) Let $X_1, X_2, \dots \in L^2$ be uncorrelated (i.e. $E[X_i X_j] = E[X_i]E[X_j]$ for $i \neq j$) with $EX_i = \mu_i$, $\mathrm{Var}(X_i)/i \to 0$ as $i \to \infty$. Let $S_n = X_1 + \dots + X_n$ and $\nu_n = ES_n/n = (\mu_1 + \dots + \mu_n)/n$. Show that $S_n/n - \nu_n \to 0$ in L^2 and in probability.

2. (Like Durrett 2.2.3) This problem deals with the idea of **Monte Carlo integration**, a technique for computing integrals by random sampling. It is named after the city in Monaco which is famous for its casino.

   (a) Let f be a measurable function on $[0,1]$ with $\int_0^1 |f(x)|dx < \infty$, let $U_1, U_2, \dots$ be iid with the uniform distribution on $[0,1]$, and let

   $$I_n = \frac{1}{n}(f(U_1) + \dots + f(U_n)).$$

   Show $I_n \to I := \int_0^1 f(x)dx$ almost surely.

   (b) Suppose $\int_0^1 |f(x)|^2 dx < \infty$. Use Chebyshev's inequality to estimate $P(|I_n - I| > a/n^{1/2})$. (Your bound should not involve I_n.)

   (c) Suppose you wanted to use this technique to compute π using the integral

   $$\pi = \int_0^1 4\sqrt{1-x^2}\,dx.$$

   Use your estimate from part 2b to determine how many samples would be needed in order to compute π to 10 decimal places with 99% confidence. I.e., find n such that $P(|I_n - I| > 10^{-10}) \leq 10^{-2}$. This may give you a feel for why Alan Sokal said: "Monte Carlo is an *extremely* bad method; it should be used only when all alternative methods are worse."

3. (Like Durrett 2.5.2) Let $X_1, X_2, \dots$ be iid. Show that if S_n/n converges almost surely, then the X_i are integrable, i.e. $E|X_i| < \infty$. (And hence by the SLLN $S_n/n \to E[X_1]$.) Hint: Show that $X_n/n \to 0$ almost surely. Decide what this says about $\sum_{n=1}^{\infty} P(|X_n| \geq n)$. Then recall a theorem (mostly) proved in class, that

   $$E|X| - 1 \leq \sum_{n=1}^{\infty} P(|X| \geq n) \leq E|X|$$

4. Problems 4 and 5 have been deferred to next week.