Homework 6: Math 6710 Fall 2010

Due in class on Friday, October 8.
Some notation:

- $\text{Var}(X) := E[(X - EX)^2] = E[X^2] - E[X]^2$
- $x \wedge y := \min(x, y)$
- $x \vee y := \max(x, y)$

1. (Durrett 5.1.9) Let $\text{Var}(X|\mathcal{G}) = E[X^2|\mathcal{G}] - E[X|\mathcal{G}]^2$. Show that

$$\text{Var}(X) = E[\text{Var}(X|\mathcal{G})] + \text{Var}(E[X|\mathcal{G}]).$$

   (This is easy but useful for the next problem.)

2. (Durrett 5.1.10) Let $Y_1, Y_2, \ldots$ be iid with mean μ and variance σ^2, N a positive integer valued random variable which is independent of the Y_n and has $EN^2 < \infty$, and let $X = Y_1 + \cdots + Y_N$. Show that X is integrable, that $EX = \mu EN$ and $\text{Var}(X) = \sigma^2 EN + \mu^2 \text{Var}(N)$.

   Suggestion: First show it under the assumption that $N \leq k$ for some k. Notice in this case that $X = \sum_{n=1}^{k} X1_{\{N=n\}}$ and think about conditional expectation. For the general case, let $X_k = Y_1 + \cdots + Y_{N \wedge k}$ and let $k \to \infty$.

3. (Durrett 5.2.6) Let $\xi_1, \xi_2, \ldots$ be independent with $E\xi_m = 0$ and $\text{Var}(\xi_m) = \sigma_m^2 < \infty$, let $S_n = \xi_1 + \cdots + \xi_n$, and let $s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$. Then $X_n = S_n^2 - s_n^2$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$.

4. (Durrett 5.2.8) If X_n, Y_n are submartingales with respect to $\mathcal{F}_n$, then so is $X_n \vee Y_n$.

5. (Durrett 5.2.13) Suppose $X_n^{(1)}$ and $X_n^{(2)}$ are supermartingales with respect to $\mathcal{F}_n$, and N is a stopping time such that $X_N^{(1)} \geq X_N^{(2)}$. Then

$$Y_n = X_n^{(1)} 1_{\{N > n\}} + X_n^{(2)} 1_{\{N \leq n\}}$$
$$Z_n = X_n^{(1)} 1_{\{N \geq n\}} + X_n^{(2)} 1_{\{N < n\}}$$

   are supermartingales. (So switching from one unfavorable game to another, even at, or right after, a strategically chosen time, doesn't help you.)