Homework 7: Math 6710 Fall 2010

Due in class on Friday, October 22.
Let $\{\mathcal{F}_n\}_{n=0}^{\infty}$ be a filtration on $(\Omega, \mathcal{F}, P)$.

1. Recall that a nonnegative integer-valued random variable N is a stopping time provided that $\{N = n\} \in \mathcal{F}_n$ for all $n \geq 0$.
   (a) If N, M are stopping times, then so is $N + M$.
   (b) Show that N is a stopping time iff $\{N \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$. This may be useful in tbe next two parts.
   (c) If N_k, $k = 1, 2, \dots$ is a sequence of stopping times and $N = \sup_k N_k$, then N is also a stopping time. (Useful special cases: $N_1 \vee N_2$ is a stopping time; and if $N_k \uparrow N$ then N is a stopping time.)
   (d) Likewise, show that $N' = \inf_k N_k$ is a stopping time.

2. Some uniform integrability examples. Recall the definition:

   Definition. A set $\{X_j : j \in J\}$ of random variables is uniformly integrable if

$$\lim_{M\to\infty} \sup_{j\in J} E[|X_j|; |X_j| > M] = 0$$

   and the alternate characterization proved in class:

   Theorem. *A set $\{X_j : j \in J\}$ of random variables is uniformly integrable if and only if the following two conditions both hold:*

   (A) (Uniform absolute continuity) For every $\epsilon > 0$ there exists $\delta > 0$ such that for all events A with $P(A) < \delta$, we have $E[|X_j|; A] < \epsilon$ for all $j \in J$;
   (B) (L^1 boundedness) $\sup_{j\in J} E[|X_j|] < \infty$.

   Now prove:

   (a) If $\{X_j : j \in J\}$ are identically distributed and integrable, then $\{X_j : j \in J\}$ is uniformly integrable.
   (b) If $\{X_i, i = 1, 2, \dots\}$ is uniformly integrable, and $S_n = X_1 + \cdots + X_n$, then $\{\frac{S_n}{n}, n = 1, 2, \dots\}$ is uniformly integrable. (In particular, the classical strong or weak law of large numbers implies L^1 convergence.)
   (c) If $\{X_j : j \in J\}$ is uniformly integrable, and $\{Y_j : j \in J\}$ is also uniformly integrable, then so is $\{X_j + Y_j : j \in J\}$.

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3. (Durrett 5.2.9) Let $\xi_1, \xi_2, \dots$ be nonnegative iid random variables with $E[\xi_i] = 1$, and suppose that $P(\xi_i = 1) < 1$ (i.e. they are not constant). Set $X_n = \prod_{i=1}^n \xi_i$.

   (a) Show that X_n is a martingale and conclude that X_n converges almost surely to some random variable X.

   (b) To determine what X is, show that $E[\sqrt{\xi_i}] < 1$ (hint: $\phi(x) = x^2$ is strictly convex) and hence $E[\sqrt{X_n}] \to 0$. Conclude that $X = 0$.

   (c) (Bonus problem) For an alternative proof, use the strong law of large numbers to show $\frac{1}{n} \ln X_n \to E[\ln \xi_i] < 0$ (hint: $\phi(x) = e^x$ is strictly convex). (Note that $E[\ln \xi_i] < +\infty$ but we could have $E[\ln \xi_i] = -\infty$; handling this case needs a little extra work.)

   Note this shows that although we know expectations of finite products of independent random variables factor, i.e. $E\left[\prod_{i=1}^n \xi_i\right] = \prod_{i=1}^n E\xi_i$, this does not carry over to infinite products.

   Interpretation: suppose ξ_i is the amount of money you end up with after betting \$1 on a game. $E[\xi_i] = 1$ so the game is fair. The strategy described by X_n is to start with \$1 and bet your entire stake on every play. We've shown that this strategy will cause you to go broke in the long run.

4. (Durrett 5.2.11) Let X_n, Y_n be positive, integrable, and adapted to $\{\mathcal{F}_n\}$. Suppose that

$$E[X_{n+1}|\mathcal{F}_n] \le (1 + Y_n) X_n$$

   for all n, and that $\sum_{n=0}^{\infty} Y_n < \infty$ a.s. (So X_n is not a supermartingale, but it's close.) Show that X_n converges a.s. to a finite limit. (Hint: Construct a related supermartingale. If you get stuck, ask me for another hint.)