Homework 8: Math 6710 Fall 2010

Due in class on Friday, October 29.

1. (a) Let $X_1, X_2, \dots$ be independent with $E[X_n] = 0$. Prove the Kolmogorov convergence criterion: if $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges almost surely. (Hint: L^2 bounded martingales.)

   (b) Let $\xi_1, \xi_2, \dots$ be an iid sequence with $P(\xi_n = 1) = P(\xi_n = -1) = \frac{1}{2}$ (fair coin flips). Show that $\sum_{n=1}^{\infty} \frac{\xi_n}{n}$ converges almost surely.

   Remark: Recall that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges, while the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges (to $\ln 2$). Although it may seem like the conditional convergence that occurs in the latter case is pretty delicate, this problem shows that if you choose signs at random, you'll get convergence unless you are really unlucky. For further reading on the random harmonic series, see B. Schmuland, "Random Harmonic Series," American Mathematical Monthly 110(5) pp. 407–416. May 2003, also available at `http://www.stat.ualberta.ca/people/schmu/preprints/rhs.pdf`.

2. (Durrett 5.5.2) Let $Z_1, Z_2, \dots$ be iid integrable random variables **with** $E[Z_i] = 0$, let θ be an integrable random variable which is independent of all the Z_n, and let $Y_n = \theta + Z_n$. Show that $E[\theta | Y_1, \dots, Y_n] \to \theta$ a.s.

   Idea: we would like to observe θ but it comes with some noise Z_n every time we observe it. However, our best guess $E[\theta | Y_1, \dots, Y_n]$ will converge to the true value of θ.

   Bonus problem (only look at this if you have nothing better to do): I suspect that $E[\theta | Y_1, \dots, Y_n] = \frac{1}{n}(Y_1 + \dots + Y_n)$, i.e. the best guess is the average of our observations, but I am not quite sure. Prove or disprove. (Update: This is false; there is a simple counterexample. I think there is a different sense in which it is the "best guess" but need to think more to make this precise.)

3. (Durrett 5.5.5 and 5.5.6)

   (a) Let X_n be nonnegative random variables. Let $D = \{X_n = 0 \text{ for some } n\}$. Suppose that for each $x \geq 0$ there exists $\delta(x) > 0$ such that for all n, $P(D | X_1, \dots, X_n) > \delta(x) > 0$ a.s. on the event $\{X_n \leq x\}$. Use the Lévy 0-1 law to conclude that $P(D \cup \{X_n \to +\infty\}) = 1$.

   (b) Let Z_n be a branching process with $p_0 := P(\xi_i^n = 0) > 0$, i.e. each individual has a positive probability of dying childless. Use the previous part to show $P(\{Z_n \to 0\} \cup \{Z_n \to +\infty\}) = 1$. So the population must either become extinct or explode; it cannot stabilize.

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4. (Some useful properties of conditional expectation)

   (a) (Useful for the next part.) Let X, Y be two integrable random variables with $EX = EY$. Show that $\mathcal{L} := \{A \in \mathcal{F} : E[X; A] = E[Y; A]\}$ is a λ-system. (I sketched this argument in class, and have proved very similar statements before.

   (b) Let X be an integrable random variable, let $\mathcal{G}_1, \mathcal{G}_2$ be σ-fields, and let $\mathcal{G} = \sigma(\mathcal{G}_1, \mathcal{G}_2)$ be the σ-field generated by both of them. Suppose that $\sigma(X, \mathcal{G}_1)$ is independent of $\mathcal{G}_2$. Show that $E[X|\mathcal{G}] = E[X|\mathcal{G}_1]$ a.s. That is, adding the "irrelevant" information $\mathcal{G}_2$ does not affect your best estimate of X.

       Hint: Use uniqueness of conditional expectation. Show that $\mathcal{P} := \{A \cap B : A \in \mathcal{G}_1, B \in \mathcal{G}_2\}$ is a π-system with $\sigma(\mathcal{P}) = \mathcal{G}$, that $\mathcal{L} := \{C \in \mathcal{F} : E[X; C] = E[E[X|\mathcal{G}_1]; C]\}$ is a λ-system, and that $\mathcal{P} \subset \mathcal{L}$.

   (c) Give an example of a random variable X and σ-fields $\mathcal{G}_1, \mathcal{G}_2$ such that X is independent of $\mathcal{G}_2$, and $\mathcal{G}_1$ is independent of $\mathcal{G}_2$, but $\sigma(X, \mathcal{G}_1)$ is not independent of $\mathcal{G}_2$ and $E[X|\mathcal{G}] \neq E[X|\mathcal{G}_1]$.

5. (Another useful property) Let $\mathbf{X}$ be a d-dimensional random vector, let $f, g : \mathbb{R}^d \to \mathbb{R}$ be Borel measurable functions, and suppose $E|f(\mathbf{X})| < \infty$. By the Doob-Dynkin lemma, we know there exists a measurable function $h : \mathbb{R} \to \mathbb{R}$ such that

   $$E[f(\mathbf{X})|g(\mathbf{X})] = h(g(\mathbf{X})) \text{ a.s.}$$

   Suppose $\mathbf{Y}$ is another random vector, and $\mathbf{X}$ and $\mathbf{Y}$ are identically distributed. Show that

   $$E[f(\mathbf{Y})|g(\mathbf{Y})] = h(g(\mathbf{Y})) \text{ a.s.}$$

   That is, the function h essentially depends only on the distribution of $\mathbf{X}$.