Homework 9: Math 6710 Fall 2010

Due in class on Friday, November 5.

1. Let $\{\mathcal{F}_n : n \geq 0\}$ be a filtration. If N is a stopping time, we define

$$\mathcal{F}_N := \{A \in \mathcal{F} : A \cap \{N = n\} \in \mathcal{F}_n \quad \forall n \geq 0\}.$$

   $\mathcal{F}_N$ can be thought of as the information available when the stopping time N arrives. We (will) have shown in class that $\mathcal{F}_N$ is a σ-field, and that $N, X_N \in \mathcal{F}_N$ where X_n is an adapted process.

   (a) (Durrett 4.1.6) If $M \leq N$ are stopping times, show $\mathcal{F}_M \subset \mathcal{F}_N$.

   (b) (Like Durrett 4.1.7) Prove the following "switching principle" for stopping times. Let K, L, M be stopping times with $K \leq L$, $K \leq M$, let $A \in \mathcal{F}_K$ be an event, and set

$$N = L1_A + M1_{A^C}.$$

   Show that N is a stopping time. (Idea: The recipe for stopping at time N is: wait until time K, check whether the event A has occurred, and depending on the answer, wait either for L or for M (neither of which will have arrived by time L)).

2. (Durrett 5.7.3) Suppose $\xi_1, \xi_2, \ldots$ are iid random variables with $P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$ (fair coin flips). Let $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$, let $S_n = \xi_1 + \cdots + \xi_n$ be a symmetric simple random walk, and for $a \in \mathbb{Z}$, let $T_a = \inf\{n : S_n = a\}$ be the hitting time of a.

   (a) For $a \neq 0$, we have previously argued that $T_a < \infty$ a.s. Show that $ET_a = \infty$. (Hint: Theorem 5.7.5.) If you are betting on a game in a fair casino with a finite bankroll and without varying your bets, you'll eventually go broke, but on the average you'll be served infinitely many complimentary cocktails along the way.

   (b) Let $a < 0 < b$ be integers, and let $N = T_a \wedge T_b$. Use it to compute $P(T_a < T_b)$, the probability that you reach a before b.

   (c) Homework 6, Problem 3 shows that $S_n^2 - n$ is a martingale. Use this fact to compute EN. (Be careful not to assume that $EN < \infty$.)

3. Consider instead an asymmetric simple random walk, with $P(\xi_i = 1) = p \in (\frac{1}{2}, 1)$, $P(\xi_i = -1) = 1 - p$. Using Homework 6, Problem 3 again, compute $\mathrm{Var}(T_b)$ for $b > 0$.

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4. Consider a random walk where $\xi_1, \xi_2, \ldots$ are iid with some arbitrary, nonconstant, integrable distribution. Suppose there exists a number $\theta < 0$ such that $E[\exp(\theta \xi_i)] = 1$.

   (a) Show that this implies $E[\xi_i] > 0$, so that the process is biased towards increasing.

   (b) Let $a < 0$ and let $T_a = \inf\{n : S_n \leq a\}$. Prove that $P(T_a < \infty) \leq \exp(-a\theta)$. (Hint: Observe that $X_n = \exp(\theta S_n)$ is a positive martingale. Express the event $\{T_a < \infty\}$ in terms of X_{T_a}.)

   (c) Show that $\liminf S_n > -\infty$ a.s. (Hint: Borel-Cantelli.)

   (d) If ξ_i has a normal distribution $N(\mu, \sigma^2)$ with mean $\mu > 0$ and variance σ^2, find θ such that $E[\exp(\theta \xi_i)] = 1$. (Recall that ξ_i has density function $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-(x-\mu)^2/2\sigma^2)$. Hint: complete the square.)

   (e) An insurance company initially has $A_0 = 10$ million dollars in assets. Its net income (premiums received minus claims paid) ξ_i in year i is normally distributed with mean $\mu = 1$ million dollars and standard deviation $\sigma = 2$ million dollars, and independent from year to year. Use the previous parts to bound the probability that the company goes bankrupt (sees its assets drop below 0).