Homework 10: Math 6710 Fall 2010

Due in class on Friday, November 12.

1. (Problem 3 from last week, now with an outline.) Consider an asymmetric simple random walk: $\xi_1, \xi_2, \ldots$ are iid with $P(\xi_i = 1) = p \in (\frac{1}{2}, 1)$, $P(\xi_i = -1) = 1 - p$; $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$, $S_n = \xi_1 + \cdots + \xi_i$. Write $\mu = E[\xi_i] = 2p - 1$, $\sigma^2 = \mathrm{Var}(\xi_i) = 4p(1-p)$. For an integer $b > 0$, let $T_b = \inf\{n : S_n = b\}$ be the hitting time of b. We wish to compute $\mathrm{Var}(T_b)$.

   (a) Set $X_n = (S_n - n\mu)^2 - n\sigma^2$. Observe by Homework 6, problem 3, that X_n is a martingale. Argue that $E[X_{T_b \wedge n}] = 0$ and so

   $$E[(S_{T_b \wedge n} - (T_b \wedge n)\mu)^2] = \sigma^2 E[T_b \wedge n] \tag{1}$$

   for all n.

   (b) Use Fatou's lemma on (1) to show $E[T_b^2] < \infty$.

   (c) Let $L = \inf\{S_n : n \geq 0\}$ be the least value achieved by S_n. Using our computation of $P(T_a < \infty)$ for $a < 0$, show that $E[L^2] < \infty$.

   (d) Use L and T_b to construct an integrable dominating function for $(S_{T_b \wedge n} - (T_b \wedge n)\mu)^2$. Then pass to the limit in (1) and use it to compute $\mathrm{Var}(T_b)$.

2. Let X_n be an L^2 martingale with $X_0 = 0$ (not necessary but saves trouble), and suppose X_n has bounded increments, i.e. for some $K < \infty$ we have $|X_{n+1} - X_n| \leq K$ a.s. for all n. Let $X_n^2 = M_n + A_n$ be the Doob decomposition of X_n^2, and let $A_\infty = \lim A_n$ (which exists a.s. since A_n is increasing). Prove $A_\infty < \infty$ a.s. on the event that X_n is bounded, i.e. $P(\{A_\infty = \infty\} \cap \{\sup_n |X_n| < \infty\}) = 0$.

   (In particular, $A_\infty < \infty$ a.s. on the event that X_n converges to a finite limit. This is a partial converse to a theorem proved in class.)

   Sketch: For $r > 0$, let $N_r = \inf\{n : |X_n| > r\}$ be the first exit time of $[-r, r]$. Use optional stopping and convergence theorems to show $E[A_{N_r}] < \infty$, and conclude $P(\{A_\infty = \infty\} \cap \{N_r = \infty\}) = 0$. Now observe that $\bigcup_{r \in \mathbb{N}} \{N_r = \infty\} = \{\sup_n |X_n| < \infty\}$.

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3. (Converging together lemmas)

   (a) Suppose $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, where c is a constant. Show $X_n + Y_n \Rightarrow X + c$.

   (b) Under the same assumptions, show $X_n Y_n \Rightarrow cX$. (To make life easier, you may just prove the special case where $Y_n \geq 0$ and $c > 0$.)

   (c) Give an example of random variables with $X_n \Rightarrow X$, $Y_n \Rightarrow Y$ (Y not constant) but $X_n + Y_n \not\Rightarrow X + Y$.

   (d) Likewise, give an example where $X_n Y_n \not\Rightarrow XY$.

4. If F, G are distribution functions, define the **Lévy distance** between them by

$$\rho(F, G) := \inf\{\epsilon \geq 0 : F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon \text{ for all } x \in \mathbb{R}\}.$$

   (a) Show that this defines a metric on the set of all distribution functions. That is, show that:

      i. $\rho(F, G) = 0$ iff $F = G$

      ii. $\rho(F, G) = \rho(G, F)$

      iii. $\rho(F, H) \leq \rho(F, G) + \rho(G, H)$ for all distribution functions F, G, H.

   (b) Show that $F_n \Rightarrow F$ iff $\rho(F_n, F) \to 0$. Thus, the topology of weak convergence is metrizable, and everything you know about metric spaces applies to it.