Homework 11: Math 6710 Fall 2010

Due in class on Wednesday, November 24, or prior to that in 593 Malott (put under door if I am out).

1. Show that $X_n \Rightarrow X$ iff $E[g(X_n)] \to E[g(X)]$ for all *compactly supported* continuous functions $g : \mathbb{R} \to \mathbb{R}$. (Recall g is compactly supported iff there exists M with $g(x) = 0$ for all $|x| > M$.) Hint: Tightness.

   Remark: In functional analysis language, this says that weak convergence of probability measures is the same as weak-* convergence in $C_c(\mathbb{R})^*$. This space is considerably nicer than $C_b(\mathbb{R})^*$. ($C_c(\mathbb{R})^*$ consists exactly of finite complex measures on $\mathbb{R}$, by the Riesz representation theorem; $C_b(\mathbb{R})$ contains some stranger objects such as finitely additive measures which require the axiom of choice to discuss.) More generally, this fact holds for random variables taking their values in a complete, separable, *locally compact* metric space M. (If M is not locally compact, $C_c(M)$ may be too small to be useful.)

2. Show that a family of random variables that is uniformly integrable is tight. Give an example of a tight family that is not uniformly integrable.

3. Recall from last week that the **Lévy metric** on the set $\mathcal{D}$ of all distribution functions on $\mathbb{R}$ is defined by

   $$\rho(F, G) := \inf\{\epsilon \geq 0 : F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon \text{ for all } x \in \mathbb{R}\}.$$

   You have showed that $(\mathcal{D}, \rho)$ is a metric space, and that convergence in the ρ metric is equivalent to weak convergence.

   (a) Suppose that F_n is a sequence of distribution functions which is Cauchy in the ρ metric. (Recall this means: for every $\epsilon > 0$ there exists N such that for all $n, m \geq N$ we have $\rho(F_n, F_m) < \epsilon$.) Show that F_n is tight and conclude that F_n converges weakly. Hence, $(\mathcal{D}, \rho)$ is a complete metric space.

   (b) Show that $(\mathcal{D}, \rho)$ is separable, i.e. contains a countable dense subset.

4. Suppose $X_n \Rightarrow X_\infty$, $Y_n \Rightarrow Y_\infty$, and for each $n \leq \infty$, X_n and Y_n are independent. (Here X_n, Y_n are all defined on the same probability space.) Prove in two different ways that $X_n + Y_n \Rightarrow X_\infty + Y_\infty$:

   (a) Using Skorohod's theorem (3.2.2). It may be helpful to think about the construction of the random variables in its proof.

   (b) Using characteristic functions and the continuity theorem (3.3.6).