Homework 12: Math 6710 Fall 2010

Due in class on Friday, December 3.

1. Let X be a random variable taking integer values, and let φ be its characteristic function.
   (a) Show that φ is 2π-periodic, i.e. $\varphi(t+2\pi) = \varphi(t)$ for all t.
   (b) Use the previous part to show that $\int_{-\infty}^{\infty} |\varphi(t)|\, dt = \infty$.
   (c) The Fourier inversion formula we proved in class does not apply to X, since φ is not integrable and X does not have a density. However, show that for any $k \in \mathbb{Z}$, we have
   $$P(X = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \varphi(t)\, dt.$$

2. Let $X_1, X_2, \dots$ be a sequence of random variables with chfs $\varphi_1, \varphi_2, \dots$, and let $p_1, p_2, \dots$ be a sequence in $[0,1]$ with $\sum_{k=1}^{\infty} p_k = 1$. Show that $\varphi(t) = \sum_{k=1}^{\infty} p_k \varphi_k(t)$ is a chf, by exhibiting a random variable X whose chf it is.

3. A function $\psi : \mathbb{R} \to \mathbb{C}$ is called **non-negative definite** if for any $t_1, \dots, t_n \in \mathbb{R}$ and any $c_1, \dots, c_n \in \mathbb{C}$, we have
   $$\sum_{r,s=1}^{n} c_r \overline{c_s} \psi(t_r - t_s) \ge 0.$$
   (This says that if we let $a_{rs} = \psi(t_r - t_s)$, then the $n \times n$ complex matrix (a_{rs}) is non-negative definite.) Show that any characteristic function φ is non-negative definite.

   This fact has a converse: every continuous, non-negative definite function $\psi : \mathbb{R} \to \mathbb{C}$ with $\psi(0) = 1$ is the characteristic function of some random variable. This is called **Bochner's theorem**, and it tells us precisely which functions are chfs.

4. (Durrett 3.4.5) Let $X_1, X_2, \dots$ be iid with mean 0 and variance $\sigma^2 \in (0, \infty)$. Let $S_n = X_1 + \cdots + X_n$, and let $Q_n = X_1^2 + \cdots + X_n^2$. Show that $S_n / \sqrt{Q_n} \Rightarrow N(0,1)$.

5. Suppose X, Y are iid with mean 0 and variance 1. Show that X, Y are $N(0,1)$ iff $\frac{X+Y}{\sqrt{2}} \overset{\mathrm{d}}{=} X \overset{\mathrm{d}}{=} Y$. (Try using chfs for one direction, and the central limit theorem for the other.)

6. Let $X_1, X_2, \dots$ be iid with mean μ and variance $\sigma^2 \in (0, \infty)$. Let $\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$ (statisticians call this the **sample mean**). Let $g : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at μ and with $g'(\mu) \neq 0$. Show that:
   $$\sqrt{n} \left(\frac{g(\bar{X}_n) - g(\mu)}{\sigma g'(\mu)} \right) \Rightarrow N(0,1).$$
   In other words, the distribution of $g(\bar{X}_n)$ is approximately $N(g(\mu), \sigma^2 g'(\mu)^2/n)$. Notice that $g(x) = x$ is the central limit theorem. This establishes that not only is $\bar{X}_n$ approximately normally distributed for large n ("**asymptotically normal**"), but so is any reasonable function of it. For reasons which I have never understood, statisticians call this fact the **delta method**.