Homework 6: Math 6710 Fall 2012

Due in class on Thursday, October 4.

1. Let $\mu_1, \mu_2, \ldots, \mu$ be probability measures on $\mathbb{R}^d$. Suppose that for every continuous $f : \mathbb{R}^d \to \mathbb{R}$ *with compact support*, we have

$$\int f \, d\mu_n \to \int f \, d\mu. \tag{1}$$

   Then $\mu_n \to \mu$ weakly, i.e. (1) also holds for *all* bounded continuous f. (Hint: Start by showing that $\{\mu_n\}$ is tight. You could then either proceed directly, or use Prohorov's theorem and a double subsequence trick.)

   This also holds on any locally compact separable metric space; if you like you could prove it in that context.

2. If μ, ν are probability measures on $\mathbb{R}^d$, their **convolution** is the probability measure $\mu * \nu$ defined by

$$(\mu * \nu)(B) = \iint 1_B(x + y) \, \mu(dx) \, \nu(dy). \tag{2}$$

   (a) Verify that $\mu * \nu$ is indeed a probability measure.
   (b) For any bounded measurable f, $\int f \, d(\mu * \nu) = \iint f(x + y) \, \mu(dx) \, \nu(dy)$.
   (c) If $X \sim \mu$, $Y \sim \nu$, and X, Y are independent, then $X + Y \sim \mu * \nu$.
   (d) If $\mu_n \to \mu$ weakly then $\mu_n * \nu \to \mu * \nu$ weakly.
   (e) (Bonus problem) If $\mu_n \to \mu$ weakly and $\nu_n \to \nu$ weakly, then $\mu_n * \nu_n \to \mu * \nu$.

3. Suppose X_n, Y_n are random variables (not necessarily independent) and we have $X_n \to X$ weakly and $Y_n \to c$ in probability.

   (a) Show that $X_n + Y_n \to X + c$ weakly. (Sometimes called Slutsky's theorem. Hint: Use Problem 1 and the fact that compactly supported continuous functions are uniformly continuous.)
   (b) Show that $X_n Y_n \to cX$ weakly.
   (c) Suppose instead that $X_n \to X$ weakly and $Y_n \to Y$ weakly, where Y need not be constant. Show that we need not have $X_n + Y_n \to X + Y$ weakly.