

RESEARCH STATEMENT

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ABSTRACT. My research lies mostly at the intersection of differential geometry and integrable systems. The primary focus of my work has been on geometric transformations between special classes of surfaces. Schematically, these transformations take one “seed” solution to a differential equation and from it generate an infinite family of solutions. Most recently I have developed a framework which simultaneously generalizes and extends several classical and modern examples of such transformations — most importantly the classical **Lie-Bäcklund transformation** of pseudospherical surfaces, the **conjugate transformation** for minimal surfaces, and the **Lawson-Bonnet transformation** relating minimal surfaces in S^n to constant mean curvature surfaces in E^n . The central result is described in section 2.2.

1. BACKGROUND

1.1. Line Congruences and Bianchi’s Theorem. In the classical differential geometry of the late nineteenth century there was much work done on the subject of *line congruences*, which are 2-parameter families of lines in E^3 . The fundamental theorem of line congruences states

Theorem 1.1. *To a generic line congruence \mathcal{L} in E^3 , there exist exactly two surfaces (called the focal surfaces) which are tangent to \mathcal{L} .*

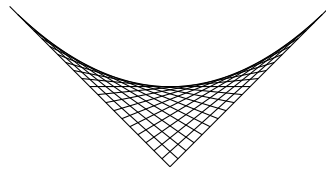


FIGURE 1. A 1-parameter family of lines in the plane. The apparent parabola along the top is the focal curve of the family. Theorem 1.1 is an extension of this idea to 2-parameter families of lines in Euclidean space.

Bianchi [2] initiated the study of line congruences with the property that the line segments connecting the two focal surfaces have constant length 1.

Theorem 1.2 (Bianchi, 1879). *Suppose \mathcal{L} is a line congruence such that the corresponding points on the two focal surfaces are at a constant distance 1. Then both of the focal surfaces are pseudospherical¹ (constant Gauss curvature $K = -1$).*

¹Throughout this document, “pseudospherical” will mean “of constant Gauss curvature -1 ”. The pseudosphere is an example, but far from the only one!

Bianchi’s theorem describes a geometric relationship between two surfaces which can only exist if the surfaces are each pseudospherical.

1.2. An Exhortation to the Reader. Before I continue, I hope you will take a moment to consider how you (personally) would go about constructing a surface with constant negative Gauss curvature. Naïvely, $K = -1$ is a second-order PDE which fails to even be quasilinear. You could search for solutions with a 1-parameter group of symmetries, reducing the problem to an ODE for the profile curve; after some calculation this should lead you to the pseudosphere and Dini’s surfaces. Can you construct more examples beyond these?

1.3. The Lie-Bäcklund Transformation. Soon after Bianchi’s theorem was published, Lie [16] described a partial converse.

Theorem 1.3 (Lie, 1880). *Let X be a pseudospherical surface. Then there exists a second pseudospherical surface \hat{X} and a line congruence \mathcal{L} such that X and \hat{X} are the focal surfaces of \mathcal{L} . Furthermore, \hat{X} and \mathcal{L} may be constructed from X by integrating a sequence of ODEs.*

Lie’s theorem² can be interpreted as a method for easily creating a new pseudospherical surface from a known one — you only need to solve some ODEs once you possess a “seed” pseudospherical surface. For example, we could start with the pseudosphere and apply Lie’s theorem to discover the very nontrivial $K = -1$ surface of Kuen [13], using nothing more than ODE techniques.



FIGURE 2. An application of the classical Lie-Bäcklund transformation to the pseudosphere (left) results in Kuen’s surface (right). Both surfaces have constant Gauss curvature $K = -1$.

1.4. Integrable Systems. At each point on a surface of strictly negative curvature, there are two distinct “flat” lines called the *asymptotic directions*. For surfaces with $K = -1$, the angle θ between the asymptotic directions satisfies the sine-Gordon equation

$$\theta_{xx} - \theta_{yy} = \sin \theta$$

where x and y are curvature-line coordinates. Except in some singular cases this process is reversible and the pseudospherical surface can be recovered from θ .

As a result, the geometrical constructions of Bianchi, Lie, and Bäcklund also describe transformations of the sine-Gordon equation which generate new solutions

²Three years later, Bäcklund [1] found an important 1-parameter family of extensions to Lie’s theorem. As a result, we now call the construction of theorem 1.3 a *Lie-Bäcklund transformation*. The extra parameter introduced by Bäcklund will be discussed later.

from old. This is the first step towards demonstrating the total integrability of the sine-Gordon equation, leading to its solution via the inverse scattering transform [22]. The key point is that the *total integrability of the sine-Gordon equation* is intimately entwined with the *existence of Bäcklund transformations for pseudospherical surfaces*.

It is a remarkable fact that this entwining is *not* unique to the sine-Gordon equation. Many common integrable PDE also appear when studying the geometry of special classes of surfaces. We already noted the connection between the sine-Gordon equation and pseudospherical surfaces. Two more notable examples include:

- Hasimoto [12] discovered that the nonlinear Schrödinger equation $\psi_{xx} + i\psi_t = |\psi|^2 \psi$ is equivalent to the motion of a vortex filament in an inviscid fluid.
- The Korteweg-de Vries equation $u_t = u_{xxx} - 6uu_x$ is related to a class of surface evolution equations [21], leading to what are known as *soliton surfaces*.

2. RESULTS

Because these interesting and ubiquitous integrable systems may also be understood as surfaces of a certain geometric class, and because the existence of a Bäcklund transformation is indicative of integrability, I wanted to construct tools which could discover just *which* classes of surfaces will admit an analog of the geometric Lie-Bäcklund transformation.

2.1. Geometric Exterior Differential Systems. My general idea was to restrict the focus from arbitrary PDE to those which are geometric in the following sense: solutions are unparameterized submanifolds of some homogeneous space which satisfy a system of G -invariant differential equations. Geometric PDE include the defining equations for pseudospherical surfaces, minimal surfaces, vortex filaments, Nambu-Goto strings, and so forth.

I chose to focus on geometric PDE for two reasons:

- A random geometric PDE is interesting; a random PDE probably is not.
- Geometric PDE may be described by a very small amount of linear data.

This second point is the moral content of the following lemma:

Lemma 2.1 (N-, 2006). *Suppose we are concerned with first-order³ PDE for functions of the form $f : X \rightarrow M$, where $M \cong G/H$ is a homogeneous space. Then there is a natural bijection between geometric PDE on M and $\text{Ad}^*(\mathfrak{h})$ -invariant subspaces Θ of $\mathfrak{h}^\perp \subseteq \mathfrak{g}^*$. Such a Θ will be called a geometric exterior differential system (gEDS).*

The point of this lemma is that we can combine the computational utility of exterior differential systems⁴ with the G -invariance of the host geometry to produce a terse set of data (the gEDS) describing a geometric PDE. A gEDS is a manifestly geometric computational tool which is *only* strong enough to describe geometrically meaningful

³This assumption is not as strict as it sounds; a variation on the usual prolongation \rightarrow submanifold \rightarrow pullback-of-contact-system trick works to reduce higher-order equations.

⁴Exterior differential systems originate in the work of Cartan [6] and Chern [9] on G -structures and the equivalence problem. Bryant, Griffiths, and Hsu [3] have made a case for the use of exterior differential systems as a basic tool for the analysis of G -structures.

equations. Stated differently, gEDS are an attempt at a minimal complete framework for working with geometric PDE.

2.2. Geometric Bäcklund Transformations. In my thesis, I aimed to find a wide-ranging generalization of the theorems of Bianchi and Lie. Because of the above considerations, I chose to define and work in the category of homogeneous spaces equipped with a gEDS. The line congruence appearing in Bianchi's theorem 1.1 is replaced by an atomic invariant relation.

From this setup, I have determined a way to derive a differential equation Δ^ρ from an invariant relation ρ . This leads to the main theorem of my thesis:

Theorem 2.2 (N-, 2009). *Let M be a homogeneous space with gEDS Θ , and let ρ be a symmetric relation on M . Then if Δ^ρ is \mathfrak{h} -invariant modulo Θ ,*

- (Generalized Bianchi) *If f and \hat{f} are two surfaces in M such that $f \sim_\rho \hat{f}$, then each surface satisfies the geometric differential equation $\Delta^\rho = 0$.*
- (Generalized Lie) *If f satisfies $\Delta^\rho = 0$, then we can construct a \hat{f} which is ρ -related to f . By the generalized Bianchi theorem, \hat{f} necessarily satisfies $\Delta^\rho = 0$. Furthermore, \hat{f} may be constructed from f by integrating a sequence of ODEs.*

I have applied this theorem to six common first-order geometries: the unit tangent bundles of the classical space forms E^3 , S^3 , and H^3 , as well as the Minkowski 3-space $M^{2,1}$ and the deSitter and anti-deSitter spaces dS^3 , AdS^3 . In the case of E^3 , theorem 2.2 reproduced classical results of Bianchi, Lie, Bäcklund, and Darboux [10], along with the 1997 and 1999 results by Chen and Li [7, 8]. Applied to Minkowski space, the theorem yields the 1982 results of TK Milnor [17]. For the other four geometries, the results are new.

Theorem 2.3 (N-, Darboux, Chen and Li, Milnor). *Let H and K denote the Gauss and mean curvatures in one of the six geometries listed above. Then surfaces which satisfy an affine relationship $aK + bH + c = 0$ admit a geometric Bäcklund transformation. The exact form that the transformation takes depends on which of the six geometries E^3 , S^3 , H^3 , $M^{2,1}$, dS^3 , AdS^3 is under consideration. The choice of geometry also places mild constraints on the coefficients a, b, c .*



FIGURE 3. A surface obtained by applying theorem 2.2 to the cylinder of radius 1 in Euclidean 3-space. This surface and the cylinder both satisfy the equation $2K - 2H + 1 = 0$. Note that this surface is asymptotic to the cylinder — it is a 1-soliton solution of $2K - 2H + 1 = 0$.

2.3. Weierstrass Representations and Mutations. A Weierstrass representation for a differential equation is a method for producing all solutions to the equation from a set of holomorphic data. The classical example is the Weierstrass-Enneper representation of minimal surfaces in E^3 from an arbitrary pair of holomorphic spinors [14]. The existence of a Weierstrass representation has made the theory of minimal surfaces (soap films) much easier than the closely related theory of constant mean curvature surfaces (soap bubbles).

Leveling the playing field, a Weierstrass representation for CMC surfaces has recently been developed. From a general theorem of Ruh and Vilms [18], a surface is CMC exactly when its normal map is harmonic with respect to the round metric on the sphere. Uhlenbeck [20] showed in 1989 how to construct all harmonic maps to the sphere from certain special maps to the group of loops in $SU(2)$. Then in 1998 Dorfmeister, Pedit, and Wu [11] gave a general method for constructing these special loops from a small amount of holomorphic data and an infinite-dimensional Iwasawa factorization theorem. Combined, these three results yield a Weierstrass representation for CMC surfaces which has led to much recent progress in the field.

In 2006, I built an alternate Weierstrass representation for CMC surfaces which circumvented the results of Ruh–Vilms. Instead of working with the harmonic normal maps, I observed the following:

Theorem 2.4 (N–, 2006). ⁵ *Let X be a simply connected immersed surface in E^3 with a moving point $x \in X$. The motion of x on X determines a curve \hat{x} in S^3 as follows:*

- *Recall that $S^3 \cong SU(2)$; mark a point on S^3 and call it $1 \in SU(2)$.*
- *Send the initial point $x(0)$ to the marked point 1 .*
- *If the x moves in the direction $v \in TX$, move the image of \hat{x} by an infinitesimal rotation through $|v|$ radians in the (v, N) -plane.*

This path-lifting procedure may be consistently extended to define a map $\hat{x} : X \rightarrow S^3$ if and only if X has constant, nonzero mean curvature. In this case, the resulting surface $\hat{x}(X)$ is minimal in S^3 .

This theorem provides a duality between CMC surfaces in E^3 and minimal surfaces in S^3 , which then allows a direct application of Uhlenbeck’s theorem and the DPW method. The transformation described in theorem 2.4 is clearly geometric, but is of a subjectively different character than the generalized Lie–Bäcklund transformations dealt with in theorem 2.2. Still, it can also be understood via a gEDS derived from a *mutation*⁶ of the underlying geometry.

Mutations are nicely compatible with gEDS computations, but I have not yet obtained many general theorems. However, the lower-dimensional cases are tractable.

⁵This theorem is essentially a geometric derivation of the Lawson-Bonnet correspondence [15] between surfaces of mean curvature H in the space form $M(k)$ and surfaces of mean curvature $H + 1$ in the space form $M(k - 1)$.

⁶Sharpe [19] defines a mutation between two homogeneous spaces G/H and \hat{G}/\hat{H} as a linear bijection $\mu : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ which is an \mathfrak{h} -module isomorphism, restricts to the identity on \mathfrak{h} , and is a Lie algebra isomorphism modulo \mathfrak{h} . Such maps can be thought of as inserting or deleting curvature into otherwise equivalent geometries; there are mutations $E^n \rightsquigarrow S^n \rightsquigarrow H^n$, for example.

In particular, I obtained new results by analyzing Minkowski space, obtaining a string-theoretic version of the Lawson-Bonnet transformation. This theorem gives a duality between strings in de Sitter space and strings in Minkowski space subject to an additional pressure-like force, and has a geometric interpretation very similar to theorem 2.4.

Theorem 2.5 (N-, 2007). *Let M be a time-like surface in de Sitter 3-space dS^3 . Then M is a solution to the Nambu-Goto equations if and only if there exists a cousin surface \hat{M} in Minkowski space $M^{2,1}$. The cousin surface satisfies the variational equation for a modified Nambu-Goto action.*

3. FUTURE RESEARCH

I have several research interests largely unrelated to this general program; in the interest of space and decency I will only list some ideas stemming from the work described above.

3.1. Higher-Order Geometries. Applications of theorem 2.2 to common first-order geometries like the classical space forms has led to some new Bäcklund transformations, as described in section 2.2. However, these transformations are not very “distant” from known examples. I am now looking to geometries of second order and up for more exotic examples.

Informally, first-order geometries can capture information about a surface and its tangent spaces. The differential equations Δ^β in theorem 2.2 involve a single derivative, so when applied to first-order geometries they yield second-order equations such as $aK + bH + c = 0$. On the other hand, higher-order geometries allow us to osculate higher-order objects to our surface, such as spheres of varying radii; this yields equations of third-order and higher.

I am currently working at applying my theorem 2.2 to Lie sphere geometry, a beautiful 10-dimensional second-order geometry of contact transformations on Euclidean space. This geometry has a rich but under-utilized surface theory:

- It is the natural geometry of the classical Ribacour transformation.
- It is the natural geometry of the Bäcklund transformation for Willmore surfaces, as described in Burstall’s book [4].
- In *Harmonic maps in unfashionable geometries* [5], Burstall and Hertrich-Jeromin obtain interesting results on “minimal” surfaces in Lie sphere geometry.

Once the correct gEDS is constructed, it will be straightforward to apply theorem 2.2 to construct Lie-sphere-geometric Bäcklund transformations between special classes of surfaces. In fact, the visualization of these surfaces and the construction of their defining variational principles would make an interesting REU-level research project. I am very excited to get my hands on these new classes of surfaces and understand how they connect to known integrable PDE. Of course, the hope is that they will actually correspond to *new* integrable systems!

3.2. Bäcklund Parameters and Nonlinear Superposition. One critical aspect of the Lie-Bäcklund transformation which I have overlooked in this paper is the existence of a Bäcklund parameter. The Bäcklund parameter appears when composing Bianchi’s

construction with a 1-parameter group of symmetries to the sine-Gordon equation, resulting in a 1-parameter family of pseudospherical Bäcklund transformations.

What one would really like to see is a theorem of the form

Theorem 3.1 (Generalized Existence of Bäcklund Parameters). *To any relation ρ satisfying [something], there exists a k -parameter family ρ_* of inequivalent relations which yield Bäcklund transformations for the same differential equation.*

Why would we like such a theorem? The existence of a Bäcklund parameter for the Lie-Bäcklund transformation leads to the fundamental Bianchi permutability theorem:

Theorem 3.2 (Bianchi Permutability). *Let B_u and B_v be pseudospherical Bäcklund transformations with the parameters u, v respectively, and let f define a pseudospherical surface. Then*

$$B_u B_v f = B_v B_u f$$

Furthermore, $B_u B_v f$ may be computed algebraically from $f, B_u f$, and $B_v f$.

This remarkable theorem can be considered a type of superposition principle for solutions, even though the underlying differential equation is highly nonlinear; f takes the role of 0, while $B_u f$ and $B_v f$ can be thought of as two solutions with “sum” $B_u B_v f$, which can be computed *without solving any differential equations at all*.

I have had some preliminary success in characterizing which equations in theorem 2.2 also admit a Bäcklund parameter by analyzing the “tangent space” of $H \setminus G / H$. Once such a characterization is known, I intend to work towards a generalization of the Bianchi permutability theorem which is applicable in any homogeneous space. Such a wide-ranging nonlinear superposition principle would be very interesting in its own right.

3.3. Bäcklund Invariants. Finally, one of my initial motivations for looking at geometric PDE was to discover which equations admitted a Weierstrass representation via the DPW method. Ideally, I am wishing for a set of quantities which can be computed for each geometric PDE and remain unchanged under Lie-Bäcklund transformations. With such invariants, we could determine which types of surfaces (e.g. CMC) are Bäcklund-equivalent to harmonic maps; this would yield a characterization of surfaces which are amenable to a DPW-type Weierstrass representation.

I believe that the first stepping stone towards finding such invariants is finding a generalization of theorem 2.2 based on mutations rather than relations. The proof as it stands needs modification — mutations bring in some extra curvatures which disrupt things and must be managed. I expect a mutation-based version of the theorem exists, but I can’t yet make out its exact form. The geometric Bäcklund invariants are still further back in the fog.

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