

# Rolling Connections

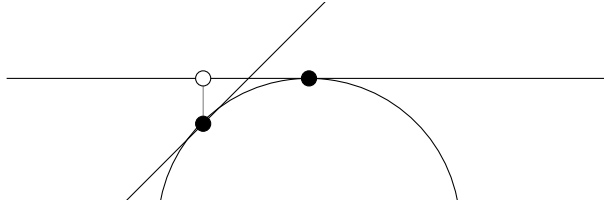
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## 1 A Geometric Model for Rolling

Let  $f : M \rightarrow \mathbb{R}^3$  define an immersed surface. We aim to formalize the notion of “rolling a plane along a curve in  $f(M)$ ”.

I claim that the following is a reasonable definition of “rolling a plane along a surface”:

**Definition 1.1.** Begin with a plane tangent to the surface  $S$  at a point  $p$ . Rolling this plane from  $p$  to  $q$  along a curve  $\gamma$  yields an isometry  $\rho_\gamma : T_p S \rightarrow T_q S$ . This rolling map is uniquely characterized by the requirement that points in any plane  $T_x S$  move in the normal direction under an infinitesimal rolling. Equivalently, we ask that all points in the plane move as little as possible as we roll.



Throughout this section, for any  $p \in \mathbb{R}^3$  and  $v \in S^2$  let us denote by  $A_{p,v}$  the affine subspace of  $\mathbb{R}^3$

$$A_{p,v} = \{p + w \mid w \in \mathbb{R}^3, w \perp N\}$$

equipped with a marked point  $p$ . Note that there are canonical isomorphisms  $A_{p,v} \cong A_{p+x,v}$  for each  $x \in v^\perp$ .

**Lemma 1.1.** *The map  $\Pi_{q,w}^{p,v} : A_{p,v} \rightarrow A_{q,w}$  determined by*

$$\Pi_{q,w}^{p,v}(p + x) = q + (p + x - q + \langle q - p - x, w \rangle w)$$

*is the orthogonal projection from  $A_{p,v}$  to  $A_{q,w}$ . The image of  $p + x$  under  $\Pi_{q,w}^{p,v}$  is the point of  $A_{q,w}$  closest to  $p + x$  in  $\mathbb{R}^3$ .*

*Proof.* To project  $p+x$  orthogonally into  $A_{q,w}$ , we must find a  $\lambda$  such that  $p+x+\lambda w \in A_{q,w}$ . This will happen precisely when

$$p + x + \lambda w - q \perp w$$

so

$$\lambda = \langle q - p - x, w \rangle$$

□

**Lemma 1.2.** *Let  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3$  and  $N : (-\varepsilon, \varepsilon) \rightarrow S^2$  be differentiable paths, and set*

$$\pi_t = \Pi_{\gamma_0, N_0}^{\gamma_t, N_t} \in \text{Hom}(A_{\gamma_t, N_t}, A_{\gamma_0, N_0})$$

*Then  $\pi'_0$  is an isometry of  $A_{\gamma_0, N_0}$ .*

*Proof.* Let  $X, Y : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3$  be vector fields along  $\gamma$  such that  $X_t$  and  $Y_t$  lie in  $A_{\gamma_t, N_t}$ , and write  $X_t = \gamma_t + x_t$ ,  $Y_t = \gamma_t + y_t$ . Now define  $D_t$  to be the squared distance between  $\pi_t(X_t)$  and  $\pi_t(Y_t)$ . Likewise, define  $d_t$  to be the squared distance between  $X_t$  and  $Y_t$ . By the previous lemma,

$$\begin{aligned} \pi_t(X_t) - \pi_t(Y_t) &= \pi_t(\gamma_t + x_t - \gamma_t - y_t) \\ &= \Pi_{\gamma_0, N_0}^{\gamma_t, N_t}(\gamma_t + x_t - \gamma_t - y_t) \\ &= \gamma_0 + (x_t - y_t - \gamma_0 - \langle x_t - y_t - \gamma_0, N_0 \rangle N_0) \end{aligned}$$

so that, writing  $v_t = x_t - y_t$ , we get

$$D_t = d_t + \langle v_t - \gamma_0, N_0 \rangle^2 - 2\langle v_t - \gamma_0, N_0 \rangle \cdot \langle v_t, N_0 \rangle$$

and

$$D'_t = d'_t - 2\langle v'_t, N_0 \rangle \cdot \langle v_t, N_0 \rangle$$

In particular, since  $v_t \perp N_t$  we have  $D'_0 = d'_0$ . Thus,  $\pi'_0$  is an isometry. □

We can use this field of projections to define a derivative of vector fields on a manifold.

**Definition 1.2.** Let  $f : M \rightarrow \mathbb{R}^3$  be a surface with normal field  $N$ ,  $v \in T_p M$ , and  $\xi$  a section of  $TM$ . Define  $\nabla_v^R \xi$  at the point  $p$  to be the unique vector in  $T_p M$  such that

$$f_p + df(\nabla_v^R \xi)_p = \frac{d}{dt} \left( \Pi_{f_p, N_p}^{f_{\gamma_t}, N_{\gamma_t}}(f_{\gamma_t} + df(\xi_{\gamma_t})) \right) \Big|_{t=0}$$

where  $\gamma : \mathbb{R} \rightarrow M$  is a curve with  $\gamma_0 = p$  and  $\gamma'_0 = v$ . We will call  $\nabla^R$  the *rolling connection* on  $M$  induced by  $f$ .

**Example 1.1.** Let

$$f(\theta, \varphi) = \begin{pmatrix} \cos \theta \cos \varphi \\ \sin \theta \cos \varphi \\ \sin \varphi \end{pmatrix}$$

parameterize the unit sphere  $S^2$  in  $\mathbb{R}^3$ . The tangent planes to  $S^2$  are of the form  $A_{x,x}$  with  $|x| = 1$ , and

$$\begin{aligned} \Pi_{x,x}^{y,y}(y+v) &= x + (y+v-x + \langle x-y-v, x \rangle x) \\ &= x + (y+v - \langle y+v, x \rangle x) \end{aligned}$$

Now pick a curve  $(\theta(t), \varphi(t))$  and a vector field  $\xi = A(\theta, \varphi) \frac{\partial}{\partial \theta} + B(\theta, \varphi) \frac{\partial}{\partial \varphi}$ . Write  $f(t)$  for  $f(\theta(t), \varphi(t))$ . Then

$$\begin{aligned} \Pi_{f(0),f(0)}^{f(t),f(t)}(f(t) + df(\xi_{f(t)})) &= f(0) + \left( f(t) - f(0) + A(f(t)) \frac{\partial f}{\partial \theta} + B(f(t)) \frac{\partial f}{\partial \varphi} \right. \\ &\quad \left. - \left\langle f(t) + A(f(t)) \frac{\partial f}{\partial \theta} + B(f(t)) \frac{\partial f}{\partial \varphi}, f(0) \right\rangle f(0) \right) \end{aligned}$$

## 2 Second Derivatives

There is some subtlety involved in forming second derivatives. In general, a (first-order) derivation on a bundle  $E \rightarrow M$  should be a map  $D : \Gamma(TM \oplus E) \rightarrow \Gamma(TE)$ . If the fibers of  $E$  are isomorphic to a manifold  $X$ , then the fibers of  $TE$  are isomorphic to  $TM \oplus TX$ . In the special case where  $X$  is a vector space, we may make use of the flatness of  $TX$  to get a canonical isomorphism  $TX = X \oplus X$ . This gives us a canonical map  $\Gamma(TE) \rightarrow \Gamma(E)$ , allowing us to think of our derivation as a map  $D : \Gamma(TM \oplus E) \rightarrow \Gamma(E)$ .

Let us make this very explicit, since similar constructions will show up as we consider bundles modeled on affine spaces or, even more generally, torsors of Lie groups. Given a vector space  $V$ , we can identify  $TV = V \oplus V$ . We can think of the projection  $(v, \xi) \mapsto \xi$  as the map which translates the vector  $v$  to the origin, thus taking  $v + \varepsilon \xi$  to  $0 + \varepsilon \xi \in T_0V$ . The critical points here are

1. The tangent bundle  $TV$  is flat.
2. There are natural maps  $T_xV \rightarrow V$ .
3. There is a distinguished point  $0 \in V$ .

The situation is slightly different in the case of an affine space  $A$  for the vector space  $V$ . We have  $TA = A \oplus V$ , so the tangent bundle is still flat. There are also natural maps  $T_x A \rightarrow A$  given by  $(x, v) \mapsto x + v$ . But  $A$  contains no distinguished points, so there is no natural choice of origin.

Let  $X, Y, Z$  be vector fields on  $M$  and  $f : M \rightarrow \mathbb{R}^n$ . We have already seen that we can differentiate vector fields using the rolling connection, so that

$$\nabla_X^R Y = X + \nabla_X Y$$

Note that

In general, a connection on a bundle  $E \rightarrow M$  is a map

$$D : \Gamma(TM \oplus E) \rightarrow \Gamma(TE)$$

which is  $C^\infty$ -linear in the  $TM$  slot and makes the diagram

$$\begin{array}{ccc} & & \Gamma(TE) \\ & \nearrow D & \downarrow \\ \Gamma(TM \oplus E) & \xrightarrow{\pi_2} & \Gamma(E) \end{array}$$

commute. If the bundle  $E$  carries extra structure, we ask that  $D$  be compatible with that structure. For example, vector bundles are equipped with bundle maps  $+$  :  $E \oplus E \rightarrow E$  and  $\cdot$  :  $\mathbb{R} \otimes E \rightarrow E$ .

### 3 Connections on Principal Bundles

Throughout this section, let  $E \xrightarrow{\pi} M$  denote a principal  $G$ -bundle on  $M$ , where  $M$  is a manifold of dimension  $n$ . We will casually write  $g \cdot \xi$  for the action of  $G$  on an element  $\xi$  of  $E$ , but when extra clarity is required, we will write  $L_g \xi$  instead.

We define a (*principal*) *connection* to be a  $n$ -plane distribution  $\Delta$  on  $E$  which is  $G$ -equivariant and lifts the identity distribution on  $TM$ . More precisely, this means that for all  $g \in G$  and  $\xi \in E$  we should have

$$\Delta_{g \cdot \xi} = dL_g(\Delta_\xi)$$

and, for all  $v \in T_p M$  and  $\xi \in \pi^{-1}(p)$ , there is a unique  $\tilde{v} \in \Delta_\xi \cap d\pi^{-1}(v)$ . From this definition, it is simple to construct things like parallel transport, curvature, and so forth.

Since distributions are somewhat harder to compute with than differential forms, we can also seek a dual description in terms of differential ideals. The distribution  $\Delta_\xi$  is dual to a unique subspace

$$\Delta_\xi^\perp = \{\phi \in T_\xi^*E \mid v \in \Delta_\xi \implies \phi(v) = 0\} \subset T_\xi^*E$$

of dimension  $\dim G$ .

The condition that  $\Delta$  be  $G$ -equivariant has the following interpretation in terms of  $\Delta^\perp$ . Let us write “ $\Delta_\xi^\perp(v) = 0$ ” as a shorthand for “ $\forall \phi \in \Delta_\xi^\perp, \phi(v) = 0$ ”. Then  $\Delta^\perp$  is  $G$ -equivariant exactly when

$$\Delta_\xi^\perp(v) = 0 \implies \Delta_{g \cdot \xi}^\perp(dL_g v) = 0$$

or, more concisely,

$$L_g^* \Delta^\perp = \Delta^\perp$$

In computations it is often handy to pick a special basis for  $\Delta^\perp$ . In particular, it would be nice to choose an ordered basis  $\{\phi_\xi^i\}_{i=0}^{\dim G}$  for each  $\Delta_\xi^\perp$  such that

$$L_g^* \phi_\xi^i = \phi_{g \cdot \xi}^i$$

Now, suppose that we have a chosen point  $\psi_p \in E_p$ . Then there is a canonical isomorphism

$$\ker d\pi \cap T_{\psi_p} E \cong \mathfrak{g}$$

given by identifying a tangent vector  $v$  with the unique  $\xi \in \mathfrak{g}$  such that

$$\left. \frac{d}{dt} L_{\exp\{t\xi\}} \psi_p \right|_{t=0} = v$$

**Lemma 3.1.** *A choice of such  $\{\phi^i\}$  on  $E_p$  determines a  $G$ -equivariant basis of the (vertical) cotangent space of  $E_p$ .*

**Lemma 3.2.** *Let  $\{\phi^i\}$  be defined over a subset  $U \subset M$  as above. Then for any such  $\phi^i$ ,*

$$d\phi^i = \sum_j \phi^j \wedge \omega_j^i$$

for some matrix of 1-forms  $\omega$ .

*Proof.* Let  $\psi^a$  be a complementary set of 1-forms, so that the  $\phi^i$  and  $\psi^a$  span  $T^*E$ . Then there exist 1-forms  $\omega_j^i$  and  $\eta_a^i$  such that

$$d\phi^i = \sum_j \phi^j \wedge \omega_j^i + \sum_a \psi^a \wedge \eta_a^i$$

We will proceed by showing that  $d\phi^i$  can be nonzero only on vertical tangent vectors. Let  $X, Y$  be vector fields on  $M$  with lifts  $\tilde{X}, \tilde{Y}$ , so that  $\Delta^\perp(\tilde{X}) = \Delta^\perp(\tilde{Y}) = 0$ . Then we have

$$d\phi^i(\tilde{X}, \tilde{Y}) = \tilde{X} \cdot \phi^i(\tilde{Y}) - \tilde{Y} \cdot \phi^i(\tilde{X}) - \phi^i([\tilde{X}, \tilde{Y}]) = 0$$

which implies

$$0 = \sum_a \psi^a(\tilde{X}) \eta_a^i(\tilde{Y}) - \psi^a(\tilde{Y}) \eta_a^i(\tilde{X})$$

Now, pick a basis  $X_a$  on  $TM$  which lifts to the dual basis  $\tilde{X}_a$  of  $\psi^a$ . Then for all  $a, b$  we have

$$0 = n_a^i(\tilde{X}_b) - n_b^i(\tilde{X}_a)$$

□

## 4 Connections on Affine Bundles

Let  $A \rightarrow M$  be a bundle of affine spaces modeled on a vector bundle  $V \rightarrow M$ . Specifically, this means that for all  $p$  in  $M$  there is an action of  $V_p$  on  $A_p$  and a corresponding canonical map

$$\ominus : A_p \oplus A_p \longrightarrow V_p$$

determined by  $\ominus(x, y) = v$  iff  $x = y + v$ .

We will frequently make use of the canonical bundle isomorphism

$$TA \cong TM \oplus A \oplus V$$

which is the affine equivalent of the vector bundle isomorphism

$$TV \cong TM \oplus V \oplus V.$$

From this, it follows that a connection on an affine bundle should be a map  $D : \Gamma(TM \oplus A) \longrightarrow TA$  of the form

$$D(X, \xi) = (X, \xi,$$

### 4.1 Curvature on Affine Bundles

Let  $\iota : TA \longrightarrow A$  be the affine map which takes the point  $(p + X, \xi) \in TA$  to  $p + X + \xi \in A$ . Then we can compute a second derivative using the rolling connection

and  $\iota$ :

$$\begin{aligned}
\iota \nabla_X^R(\iota \nabla_Y^R(f_p + Z)) &= \iota \nabla_X^R(\iota(Y + \nabla_Y Z)) \\
&= \iota \nabla_X^R(f_p + Z + Y + \nabla_Y Z) \\
&= \iota(X + \nabla_X Z + \nabla_X Y + \nabla_X \nabla_Y Z) \\
&= f_p + Z + Y + \nabla_Y Z + X + \nabla_X Z + \nabla_X Y + \nabla_X \nabla_Y Z
\end{aligned}$$

Since  $\iota \circ \nabla^R$  takes values in an affine space, we cannot compute the curvature tensor as usual; there is no way to make sense of a general sum of elements of an affine space. To remedy this we must fix a basepoint and then interpret our affine vectors as displacements from this basepoint. Luckily, there is a natural choice — since we are differentiating the affine field  $f_p + Z$ , we should think of the derivatives as living in the tangent space of  $f_p + Z$ . In other words, we should treat the second derivative above as the displacement

$$Y + \nabla_Y Z + X + \nabla_X Z + \nabla_X Y + \nabla_X \nabla_Y Z \in T_{f_p + Z} A$$

Let us write  $\delta$  for this map taking  $A$  to  $T_{f_p + Z} A$ . Then we may compute the affine curvature as

$$\begin{aligned}
R^{\text{aff}}(X, Y)Z &= \delta \iota \nabla_X^R \iota \nabla_Y^R Z - \delta \iota \nabla_Y^R \iota \nabla_X^R Z - \delta \iota \nabla_{[X, Y]}^R Z \\
&= Y + \nabla_Y Z + X + \nabla_X Z + \nabla_X Y + \nabla_X \nabla_Y Z \\
&\quad - (X + \nabla_X Z + Y + \nabla_Y Z + \nabla_Y X + \nabla_Y \nabla_X Z) \\
&\quad - ([X, Y] + \nabla_{[X, Y]} Z) \\
&= T^\nabla(X, Y) + R^\nabla(X, Y)Z
\end{aligned}$$

where  $T^\nabla$  and  $R^\nabla$  are respectively the torsion and curvature of  $\nabla$ . Of course, since the Levi-Civita connection has vanishing torsion, this simply reduces to

$$R^{\text{aff}} = R^\nabla$$

However, connections with nonvanishing torsion will become important later, so it is useful to see how torsion terms appear in the expression for affine curvature.

## 4.2 The Underlying Principal Bundle

Just as Riemannian geometry on the tangent bundle is determined by the  $O(n)$ -geometry of the frame bundle, we can understand the affine connection on the affine tangent bundle through a principal  $AO(n)$ -bundle of affine frames.

**Definition 4.1** (The Affine Frame Bundle). Let  $M$  be a manifold equipped with a Riemannian metric, and define

$$\mathcal{A}_p = \{(\xi, \beta) \mid \xi \in T_p M, \beta \text{ an ordered orthonormal basis of } T_p M\}$$

We may think of an element of  $\mathcal{A}_p$  as a point of  $T_p M$  along with an orthonormal frame attached to the point. There is a natural, free, and transitive action of  $AO(n)$  on  $\mathcal{A}_p$  which makes  $\mathcal{A}$  into a principal  $AO(n)$ -bundle. We will call  $\mathcal{A}$  the *affine frame bundle*.

The rolling connection described above extends naturally to the affine frame bundle. By rolling the entire affine frame along a curve, frames at two different points may be compared and fields of frames may be differentiated. Since rolling acts by affine motions, the derivative must act as an affine transformation. Indeed, this is what we already saw; if  $\Psi$  is a local trivialization mapping  $\mathbb{R}^n$  affinely to  $TM$  with  $\Psi(0) = f_p$  then

$$\nabla_X^R \Psi(Y) = \left( D_X + \begin{pmatrix} 0 & 0 \\ \text{id}(X) & \Gamma(X) \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ Y \end{pmatrix}$$

where  $\Gamma$  is the connection 1-form for the Levi-Civita connection in this trivialization.

## 5 Rolling and the Levi-Civita Connection

**Definition 5.1.** Recall that the *Levi-Civita connection*  $\nabla$  on an immersed manifold may be defined by

$$df(\nabla_v \xi) = \left( \frac{d}{dt} df(\xi_{\gamma_t}) \Big|_{t=0} \right)^T$$

where  $V^T$  means the tangential projection of  $V$ .

**Theorem 5.1.** *The rolling connection  $\nabla^R$  has the following properties:*

1.  $\nabla_v^R$  acts on  $T_p M$  by isometries.
2.  $\nabla_v^R$  is a derivation:  $\nabla_v^R(f\xi) = df(v)\xi + f\nabla_v^R \xi$ .
3.  $\nabla_v^R \xi = v + \nabla_v \xi$ , where  $\xi$  is the Levi-Civita connection.

*Proof.* Let us prove the properties in order:

1. This follows immediately from the previous lemma, but let us compute  $\nabla_v^R \xi$  explicitly. To simplify notation, let us write  $\pi_t$  for  $\Pi_{f_p, N_p}^{f_{\gamma_t}, N_{\gamma_t}}$  and  $\hat{\xi}_t$  for  $f_{\gamma_t} + df(\xi_{\gamma_t})$  above. Then

$$\begin{aligned}
\frac{d}{dt}\pi_t(\hat{\xi}_t) &= \frac{d}{dt} \left( f_p + (\hat{\xi}_t - f_p + \langle f_p - \hat{\xi}_t, N_p \rangle N_p) \right) \\
&= \hat{\xi}_t' - \langle \hat{\xi}_t', N_p \rangle N_p \\
&= df(v) + \frac{d}{dt} df(\xi_{\gamma_t}) - \left\langle \frac{d}{dt} df(\xi_{\gamma_t}), N_p \right\rangle N_p \\
&= df(v) + df(\nabla_v \xi)
\end{aligned}$$

□

Let us relate the curvature of the rolling connection to more well-known geometric objects. We temporarily let  $\nabla$  denote any connection on  $TM$  and  $\nabla_X^R = X + \nabla_X$ . Then the curvature of  $\nabla^R$  is

$$\begin{aligned}
\text{curv } \nabla^R(X, Y) &= \nabla_X^R \nabla_Y^R - \nabla_Y^R \nabla_X^R - \nabla_{[X, Y]}^R \\
&= \nabla_X^R(Y + \nabla_Y) - \nabla_Y^R(X + \nabla_X) - [X, Y] - \nabla_{[X, Y]} \\
&= X + \nabla_X Y + \nabla_X \nabla_Y - Y - \nabla_Y X - \nabla_Y \nabla_X - [X, Y] - \nabla_{[X, Y]} \\
&= X - Y + T^\nabla(X, Y) + R^\nabla(X, Y)
\end{aligned}$$

if  $\nabla_X^R Y = X + Y + \nabla_X Y$  then

$$\begin{aligned}
\text{curv } \nabla^R(X, Y)Z &= \nabla_X^R(Y + Z + \nabla_Y Z) - \nabla_Y^R(X + Z + \nabla_X Z) - [X, Y] - Z - \nabla_{[X, Y]}Z \\
&= X + Y + Z + \nabla_Y Z + \nabla_X Y + \nabla_X Z + \nabla_X Y \\
&\quad - Y - X - Z - \nabla_X Z - \nabla_Y X - \nabla_Y Z - \nabla_Y \nabla_X Z \\
&\quad - [X, Y] - Z - \nabla_{[X, Y]}Z \\
&= T^\nabla(X, Y) - Z + R^\nabla(X, Y)Z
\end{aligned}$$

The rolling connection gives a connection on  $TTM$ , so we should interpret  $\nabla_X^R Y$  as a tangent vector to  $Y$ ; that is, as the vector  $Y + \nabla_X^R Y \in TM$ .

$$X \cdot Y = Y + \varepsilon(X + \nabla_X Y)$$

$$X \cdot Y \cdot Z - Y \cdot X \cdot Z - [X, Y] \cdot Z = X \cdot (Z + \varepsilon(Y + \nabla_Y Z)) - Y \cdot (Z + \varepsilon(X + \nabla_X Z)) - Z - \varepsilon[X, Y] -$$

## 6 Rolling With Homogeneous Spaces

When both spaces being rolled together are homogeneous, the rolling connection takes on a still simpler form.

### 6.1 Rolling a Sphere and a Plane

As an extended example, let us work out all the details of rolling a plane along a sphere. We will parameterize the sphere by

$$f(\theta, \varphi) = \begin{pmatrix} \cos \theta \cos \varphi \\ \sin \theta \cos \varphi \\ \sin \varphi \end{pmatrix}$$

with tangent vector fields away from the poles given by

$$\frac{\partial f}{\partial \theta} = \begin{pmatrix} -\sin \theta \cos \varphi \\ \cos \theta \cos \varphi \\ 0 \end{pmatrix} \quad \frac{\partial f}{\partial \varphi} = \begin{pmatrix} -\cos \theta \sin \varphi \\ -\sin \theta \sin \varphi \\ \cos \varphi \end{pmatrix}$$

The Levi-Civita connection acts as follows:

$$\nabla_{\partial_\theta} \partial_\theta =$$