Related Rates Solutions

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1. (a) *What we know:* Let's call the distance along the ground from the truck to the pulley $T(t)$, the distance from the truck's bumper to the pulley $D(t)$, and the height of the weight off of the ground $h(t)$. The problem tells us
$$\frac{dT}{dt} = 2.5 \text{ ft/s}$$
From the Pythagorean theorem and the fact that the bumper is 2 feet off the ground, we know
$$D^2 = T^2 + 18^2$$
and the fact that the rope isn't stretching or changing length says that
$$D + (20 - h) = \text{const}$$
   (b) *What we want:* The problem is asking how fast the weight is rising when the truck is 15 feet away ($T(t) = 15$), so we want to know dh/dt.
   (c) *Wish List:* If we knew dh/dT we could get what we want since
$$\frac{dh}{dT} \cdot \frac{dT}{dt} = \frac{dh}{dt}$$
   (d) *Getting what we want:* If we use implicit differentiation on the two algebraic equations that we found, we get
$$2D\ dD = 2T\ dT$$
and
$$dD - dh = 0$$
The second equation means that $dh = dD$. Using this in the first equation, we can solve for dh/dT (what we wanted):
$$2D\ dh = 2T\ dT$$

so

$$\frac{dh}{dT} = \frac{T}{D}$$

This means that

$$\frac{dh}{dt} = \frac{dh}{dT} \cdot \frac{dT}{dt} = \frac{T}{D} \cdot 2.5 \text{ ft/s}$$

We are interested in what happens when $T = 15$:

$$\frac{dh}{dt} = \frac{15 \cdot 2.5}{D} \text{ ft/s}$$

When $T = 15$, the Pythagorean theorem tells us that $D = \sqrt{15^2 + 18^2}$, so altogether we have

$$\frac{dh}{dt} = \frac{15 \cdot 2.5}{\sqrt{15^2 + 18^2}} \text{ ft/s}$$

which is about 1.6 feet per second.

2. (a) *What we know:* Let's call the angle between the hands $\theta(t)$ and the distance between the tips $D(t)$. The law of cosines tells us that

$$D^2 = 2^2 + 3^2 + 2 \cdot 2 \cdot 3 \cdot \cos(\theta)$$

since the hour hand has length 2 and the minute hand has length 3. Since the hour hand goes around the clock once per 12 hours, it must move at a rate of $360/12 = 30$ degrees per hour, or 0.5 degrees per minute. Since the minute hand moves once around the clock per hour, it must move at a rate of 360 degrees per hour, or 6 degrees per minute. Altogether, this means that the angle between them changes like

$$\frac{d\theta}{dt} = 5.5 \text{ degrees/min} = \frac{2\pi \cdot 5.5}{360} \text{ rad/min} = \frac{11\pi}{360} \text{ rad/min}$$

so

$$\theta(t) = \frac{11\pi}{360} t \text{ min}$$

(b) *What we want:* We want to how fast the distance between the tips of the hands is changing: dD/dt.

(c) *Wish List:* If we knew $dD/d\theta$ we could get what we want by:

$$\frac{dD}{dt} = \frac{dD}{d\theta} \cdot \frac{d\theta}{dt}$$

(d) *How to get what we want:* If we differentiate the law of cosines, we get

$$2D\ dD = -12 \sin(\theta) d\theta$$

so

$$\frac{dD}{d\theta} = -6\frac{\sin(\theta)}{D}$$

Now we can solve for dD/dt:

$$\frac{dD}{dt} = \frac{dD}{d\theta} \cdot \frac{d\theta}{dt} = -6\frac{\sin(\theta)}{D} \cdot \frac{11\pi}{360} = \frac{-11\pi\sin(\theta)}{D}$$

Now all we need to know is θ and D at $2:30$. $2:30$ is 150 minutes after noon, so $t = 150$. This means that

$$\theta(150) = \frac{-11 \cdot \pi \cdot 150}{360} = \frac{-55}{12}\pi$$

and using the law of cosines equation,

$$D(150) = \sqrt{13 + 12\cos(\frac{-55}{12}\pi)}$$

Plugging these in finally gives us the answer

$$\frac{dD}{dt} = \frac{-11\pi\sin(\frac{-55}{12}\pi)}{\sqrt{13 + 12\cos(\frac{-55}{12}\pi)}}$$

3. (a) *What we know:* The surface area of the reservoir is given by πr^2 and the volume by

$$V = \frac{1}{3}\pi r^2 h$$

The problem tells us that the volume decreases by twice the surface area per week:

$$\frac{dV}{dt} = 2\pi r^2 \text{ m}^3/\text{wk}$$

We know that the reservoir is shaped like a right circular cone, so we also have

$$h = r$$

since the cross-section is a 45 degree right triangle.

(b) *What we want:* We need to find how fast the height of the water is decreasing: dh/dt

(c) *Wish List:* If we knew dh/dV then we could get what we want by

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$

(d) *How to get what we want:* The second equation means that we can replace each r with an h in the first equation, giving

$$V = \frac{1}{3}\pi h^3$$

Differentiate this to get

$$dV = \pi h^2 dh$$

so

$$\frac{dh}{dV} = \frac{1}{\pi h^2}$$

Now we can get what we want:

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{1}{\pi h^2} \cdot 2\pi r^2 \text{ m}^3/\text{wk} = 2 \text{ m}^3/\text{wk}$$

where we used $r = h$ again.

4. (a) I'll approach this problem a little more directly than the others. We should probably work in feet, so the train is moving at 528000 feet per hour. We will call the position of the train $T(t)$, so

$$\frac{dT}{dt} = 528000 \text{ ft/hr}$$

Let us figure out how fast the wheels are spinning. Since the inner wheel is the one driving the train and the inner wheel has a radius of 1.3 feet, the train moves forward $2\pi \cdot 1.3$ feet per revolution. Therefore, the wheels make

$$\frac{528000}{2\pi \cdot 1.3} \text{ revolutions/hr}$$

If $\theta(t)$ is the angle that the wheel has turned, this says that

$$\frac{d\theta}{dt} = \frac{528000}{2\pi \cdot 1.3} \text{ revs/hr}$$

A point on the bottom of the outer rim B is moving backwards at a rate of

$$\frac{dB}{dt} = -2\pi \cdot 1.5 \cdot \frac{d\theta}{dt} \text{ ft/hr}$$

so the point on the bottom of the outer rim is moving backwards at the rate

$$\frac{dB}{dt} = \frac{-2\pi \cdot 1.5 \cdot 528000}{2\pi \cdot 1.3} \text{ revs/hr} = -528000 \cdot \frac{1.5}{1.3} \text{ revs/hr}$$

relative to the axle of the train. This means that relative to the ground, the point is moving with velocity

$$\frac{dB}{dt} + \frac{dT}{dt} = 528000(-\frac{1.5}{1.3} + 1) \text{ ft/hr} = 100 \cdot (1 - \frac{1.5}{1.3}) \text{ mi/hr}$$

or about 15.4 miles per hour *backwards*!