Class Notes
2/8–2/15

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1 Connectedness

When we defined the subspace topology we noted that if $Y \subset X$ then subsets open in Y might not be open in X. Note however, that if Y itself is open in X, then everything open in Y is open in X (since it equals $Y \cap A$, where A is open in X).

Therefore, if a space X equals the disjoint union of non-empty open sets A and B, then the open sets of X are nothing more than the unions of sets open in A and sets open in B; essentially, the topology on X can be split up, or separated, into the topologies on A and B. We say that a space is disconnected if there is such a separation.

Definition 1 (Separation). Two sets A and B form a *separation* of a topological space X if A and B are disjoint, non-empty open subsets of X and $A \cup B = X$.

Definition 2 (Disconnectedness). A space X is *disconnected* if there is a separation on it.

Definition 3 (Connectedness). A space X is *connected* if it isn't disconnected.

Letting $D(p) = \{\, q \in \mathbb{R}^2 \mid |q - p| \leq 1 \,\}$, an example of a disconnected space would be $D(\langle 0,0\rangle) \cup D(\langle 2,3\rangle) \cup D(\langle -7,-5\rangle)$. Note that this space has more than one separation, for example, $A = D(\langle 0,0\rangle) \cup D(\langle 2,3\rangle)$ and $B = D(\langle -7,-5\rangle)$ or $A = D(\langle 2,3\rangle) \cup D(\langle -7,-5\rangle)$ and $B = D(\langle 0,0\rangle)$.

Definition 4 (Components). If X is a space and $A \subset X$ such that A is connected and A is not properly contained in any larger connected subset of X, then A is a *component* of X.

Definition 5 (Total Disconnectedness). If X is a space (containing more than one point) such that all the components of X are 1-point sets, X is called *totally disconnected.*

Note that if a space is connected, then it has only one component, namely, itself. Every connected subspace of a space lies in exactly one component, as can be deduced from the following theorem.

Theorem 1. *If A and B form a separation of X and Y is a connected subspace of X, either $Y \subset A$ or $Y \subset B$.*

Proof. The sets $Y \cap A$ and $Y \cap B$ are disjoint and are both open in Y (since A and B are open in X). Also, $(Y \cap A) \cup (Y \cap B) = Y$. If they were both non-empty, then they would form a separation of Y, but Y is connected. Therefore one of them must be empty (say $Y \cap B = \emptyset$), therefore $Y \cap A = Y$ and $Y \subset A$. ☐

It's obviously not the case that the union of two connected subspaces of a space is connected. However, it *is* the case that the union of two connected subspaces which aren't disjoint is connected.

Theorem 2. *Let X be a space, and U, $V \subset X$. If $U \cap V \neq \emptyset$, then $U \cup V$ is connected.*

Proof. Assume not and let A and B form a separation of $U \cup V$. $U \cap V$ is non-empty, so pick some $x \in U \cap V$. Either $x \in A$ or $x \in B$; assume $x \in A$. Then, by the previous theorem, we know that $U \subset A$ and $V \subset A$ so $B = \emptyset$ and it isn't a separation. ☐

Theorem 3. *Let X, Y be topological spaces, $f\colon X \to Y$ be continuous and surjective (onto). Then if X is connected, Y is connected.*

Proof. Assume X is connected and Y is not. Let A and B form a separation of Y. Then $f^{-1}[\![A]\!]$ and $f^{-1}[\![B]\!]$ are disjoint open sets of X such taht $f^{-1}[\![A]\!] \cup f^{-1}[\![B]\!]$ in virtue of the fact that f is continuous. They are also both non-empty in virtue of the fact that f is surjective. So X is not connected, contrary to assumption. ☐

It's an important fact that closed intervals in the real line are connected spaces.

Theorem 4. *The topological space $[a, b]$ (in the real line) is connected.*

Proof. Let A and B be a separation of $[a, b]$ and assume $a \in A$. We can also assume $b \in B$, since, if $a \in A$ there must be some $x \in [0, 1]$ with $x \in B$; then just apply an appropriate linear transformation to A and B to obtain a new separation of $[a, b]$ with $b \in B$.

Now let x_0 be the least upper bound of A. It must either be in A in B. Assume it's in B. Then we can find an interval (a', b') such that $x_0 \in (a', b') \subset B$. Also $(x_0, b] \subset B$ since x_0 is the least upper bound of A. But then $(a', b] \subset B$ and any number c such that $a' < c < x_0$ is an upper bound for A less than x_0, contradicting the assumption that x_0 was the least upper bound.

So assume x_0 is in A. But then there is an (a', b') such that $x_0 \in (a', b') \subset A$. But then x_0 isn't an upper bound for A, since for every c such that $x_0 < c < b'$ we have $c \in A$. ☐

Definition 6 (Convexity on $\mathbb{R}$). A subset $A \subset \mathbb{R}$ is *convex* if for all a, $b \in A$, $[a, b] \subset A$.

Note that such things as closed intervals, open intervals, open or closed rays, and the entire real line are convex. Based on the preceding theorem, we can prove the following.

Theorem 5. *Any convex subset of $\mathbb{R}$ is connected.*

Proof. Let C be a convex subset of $\mathbb{R}$. Let A and B be a separation of C. Pick an $a \in A$ and a $b \in B$. Then $A \cap [a, b]$ and $B \cap [a, b]$ form a separation of $[a, b]$, but we have just proven that closed intervals are connected. $\square$

The converse is also true, as one can readily see.

Theorem 6. *If a subset A of $\mathbb{R}$ isn't convex, it isn't connected.*

Proof. If A isn't convex, then there exist $a < c < b$ such that a, $b \in A$ but $c \notin A$. Then $(-\infty, c) \cap A$ and $(c, \infty) \cap A$ form a separation of A. $\square$

We can now prove the Intermediate Value Theorem of Calculus.

Theorem 7 (Intermediate Value Theorem). *Let $f \colon \mathbb{R} \to \mathbb{R}$. If $a < b$ and r is between $f(a)$ and $f(b)$ then there exists a c such that $a < c < b$ and $f(c) = r$.*

Proof. The interval $[a, b]$ is connected so $f[\![[a, b]]\!]$ is connected and a subset of $\mathbb{R}$, so it's convex. We have $f(a)$, $f(b) \in f[\![[a, b]]\!]$, so $r \in f[\![[a, b]]\!]$ (by convexity), so there's a $c \in [a, b]$ with $f(c) = r$ (by the definition of $f[\![[a, b]]\!]$. $\square$

2 Path Connectedness

In the last section we examined one formalization of the concept of "connectedness;" in this section we will examine another, perhaps more intuitive. It's the concept of a space being connected if we can "connect" any two points with a path. First, of course, we must define what we mean by a path.

Definition 7 (Path). Let X be a space and x, $y \in X$. A *path* from x to y is a continuous $f \colon [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

For "reasonable" topological spaces a path is just what you think it is. We can now define the notion of path-connectedness.

Definition 8 (Path-Connectedness). A space X is *path-connected* if for any two points x, $y \in X$ there is a path from x to y.

As a first basic theorem we have the following.

Theorem 8. *If a space is path-connected, it's connected.*

Proof. Let X be path connected and assume A and B form a separation of X. Pick $a \in A$ and $b \in B$ and find a path f from a to b. But then $f^{-1}[\![A]\!]$ and $f^{-1}[\![B]\!]$ are non-empty (since 0 is in the former and 1 in the latter), open (since f is continuous), disjoint (since A and B are disjoint) subsets of $[0,1]$ whose union is $[0,1]$. So then $[0,1]$ would be disconnected. But it isn't. $\square$

The converse is false, however. There exist connected spaces which aren't path-connected.

To construct one, start with the following subset of $\mathbb{R}^2$: $C = [0,1] \times \{0\} \cup (\bigcup\{\, \{1/n\} \times [0,1] \mid n = 1, 2, \dots \})$. (You should probably draw a picture of this set to follow the discussion.)

This set is clearly path-connected (and therefore connected) since one can get from any "spike" to any other by moving down to the x-axis, moving over and moving up. Now form the space we're interested in by adding the point $p = \langle 0, 1 \rangle$; $C' = C \cup \{p\}$. This space is still connected; let A and B be a separation. Then C must be contained within A or B (since it's connected), let's say $C \subset A$. A is closed as well as open (since $A^c = B$ is open) so it must contain all of its limit points. But p is clearly a limit point of C so $C \cup \{p\} = C' \subset A$ and $B = \emptyset$.

However it is intuitively obvious that there is no path from p to any other point of the space, as that require jumping a nonzero distance from p to the x-axis or a nonzero distance to one of the spikes.

That space is called the deleted comb space. There is another space which is connected but not path connected which uses essentially the same "trick:" it's called the topologist's sine curve and it is $[-1,1] \times \{0\} \cup \{\, \langle x, \sin 1/x \rangle \mid 0 < x < 1 \,\}$.

3 Local Formulations of Connectedness

There are two other concepts which we will use in algebraic topology.

Definition 9 (Local Connectedness). A space X is *locally connected at a point* x if for every open set U containing x there is a connected open set V with $x \in V \subset U$.

A space X is *locally connected* if it is locally connected at all of it's points.

Definition 10 (Local Path-Connectedness). A space X is *locally path-connected at a point* x if for every open set U containing x there is a path-connected open set V with $x \in V \subset U$.

A space X is *locally path-connected* if it is locally path-connected at all of its points.

Local path-connectedness implies local connectness, but a space can be locally connected without being locally path-connected. Also, local connectedness and local path-connectedness are independent of connectedness and path-connectedness respectively. Examples follow.

Add the spike over 0 to the deleted comb space to form $C'' = \{0\} \times [0,1] \cup C'$. C'' is clearly path-connected (and therefore connected) but it isn't locally connected (and therefore not locally path-connected): take $p = \langle 0, 1 \rangle$. Any small neighborhood of it will contain infinitely many disjoint intervals, so it isn't locally connected.

An example of a space which is locally connected but not locally path-connected (as opposed to the space I presented in class, which had a point such that it was locally connected at that point but not locally path-connected at that point) is $[0,1] \times [0,1]$ under the topology generated by the following basis: $\{\, \{x\} \times (a,b) \mid x,a,b \in [0,1], a < b \,\} \cup \{\, \{x\} \times [0,b) \mid x,b \in [0,1], 0 < b \,\} \cup \{\, \{x\} \times (a,1] \mid x,a \in [0,1], a < 1 \,\} \cup \{\, \langle x,y \rangle \in [0,1] \times [0,1] \mid (\exists a,b,c,d \in [0,1]) a < y < b$ or $a = y, x > c$ or $y = b, x < d \,\}$

The reader is invited to prove that this space is locally connected but not locally path-connected as an exercise. Or, as an alternative exercise, the reader is invited to wait until I build up the motivation to type it up myself.