Math 191 – Fall, 2004 – Prelim I "Solutions"

9/21/04

1a. By the Fundamental Theorem of Calculus we have

$$\frac{d}{dx}\int_0^x \cos t\ dt = \cos x.$$

1b. Let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x$. Then by the Fundamental Theorem and the Chain Rule we have

$$\frac{d}{dx}\int_0^{\sin x} t^2\ dt = \left(\frac{d}{du}\int_0^u t^2\ dt\right)\frac{du}{dx} = u^2\frac{du}{dx} = \sin^2 x\cos x.$$

2a. Let $u = 1 + x^3 \Rightarrow du = 3x^2\ dx$, so

$$\int_0^2 3x^2\sqrt{1+x^3}\ dx = \int_1^9 \sqrt{u}\ du = \left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{2}{3}(9^{3/2} - 1^{3/2}) = \frac{2}{3}(27-1) = \frac{52}{3}.$$

2b. Let $u = \sin x \Rightarrow du = \cos x\ dx$, so

$$\int_0^{\pi/2} \cos x\cos(\sin x)\ dx = \int_0^1 \cos u\ du = [\sin u]_0^1 = \sin 1.$$

3. The error formula for the trapezoidal rule is $|E_T| \le \frac{b-a}{12}h^2M$ where M is any upper bound on $|f''|$. In this problem we have $b = 2$, $a = 0$, $h = \frac{b-a}{n} = \frac{2}{n}$, and $f''(x) = 6x + 4$. Therefore, since $|f''| \le 6\cdot 2 + 4 = 16$ on $[0, 2]$ we have $M = 16$. In order to make $|E_T| < \frac{1}{100}$, we need

$$\begin{aligned} \frac{b-a}{12}h^2M = \frac{2}{12}\left(\frac{2}{n}\right)^2\cdot 16 &< \frac{1}{100}, \\ \frac{2^5}{3n^2} &< \frac{1}{2^2\cdot 5^2}, \\ n^2 &> \frac{2^7\cdot 5^2}{3}, \\ n &> \sqrt{\frac{2^7\cdot 5^2}{3}} \\ n &> 2^3\cdot 5\sqrt{\frac{2}{3}} = 40\sqrt{\frac{2}{3}}. \end{aligned}$$

Since $\sqrt{\frac{2}{3}} < 1$, we can choose $n = 40$ to make the error less than $\frac{1}{100}$.

4a. By definition, $\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\ dx$. Therefore, $\text{av}(f)$ over $[0, 2]$ is given by $\frac{1}{2}\int_0^2 f(x)\ dx$. Using one of our rules for definite integrals we have $\int_0^2 f(x)\ dx = \int_0^1 f(x)\ dx + \int_1^2 f(x)\ dx$. Since $\text{av}(f)$ over $[0, 1]$ is 8 we have $\int_0^1 f(x)\ dx = (1-0)\cdot 8 = 8$ and since $\text{av}(f)$ over $[0, 2]$ is 4 we have $\int_1^2 f(x)\ dx = (2-1)\cdot 4 = 4$. Therefore, $\text{av}(f)$ over $[0, 2]$ is $\text{av}(f) = \frac{1}{2}(8+4) = 6$.

4b. To get $\text{av}(f)$ over $[3, 6]$ we use the same reasoning as above. By definition, $\text{av}(f) = \frac{1}{6-3}\int_3^6 f(x)\ dx = \frac{1}{3}\int_3^6 f(x)\ dx$. Since $\text{av}(f)$ over $[3, 4]$ is 3 we have $\int_3^4 f(x)\ dx = (4-3)\cdot 3 = 3$ and since $\text{av}(f)$ over $[4, 6]$ is 9 we have $\int_4^6 f(x)\ dx = (6-4)\cdot 9 = 18$. Therefore, $\text{av}(f)$ on $[3, 6]$ is $\text{av}(f) = \frac{1}{3}\left(\int_3^4 f(x)\ dx + \int_4^6 f(x)\ dx\right) = \frac{1}{3}(3+18) = 7$.

5. The area of each cross section is given by $A(x) = \frac{1}{2}[s(x)]^2$ where $s(x) = (2 - x^2) - (-6 + x^2) = 8 - 2x^2$. The parabolas intersect at $x = \pm 2$ so the volume is given by:

$$\begin{aligned} \text{volume} &= \int_{-2}^{2} A(x)\ dx = \int_{-2}^{2} \frac{1}{2}(8 - 2x^2)^2\ dx, \\ &= \frac{1}{2}\int_{-2}^{2} 64 - 32x^2 + 4x^4\ dx = \frac{1}{2}\left[64x - \frac{32}{3}x^3 + \frac{4}{5}x^5\right]_{-2}^{2} = \frac{1024}{15}. \end{aligned}$$

6. To find the volume we use washers. The inner radii are given by $r(y) = \sin y$ and the outer radii are given by $R(y) = 2 - \cos y$. Therefore, the volume is:

$$\begin{aligned} \text{volume} &= \int_{0}^{\pi} \pi[R(y)^2 - r(y)^2]\ dy = \int_{0}^{\pi} \pi[(2 - \cos y)^2 - (\sin y)^2]\ dy, \\ &= \pi\int_{0}^{\pi} 4 - 4\cos y + \cos^2 y - \sin^2 y\ dy = \pi\int_{0}^{\pi} 4 - 4\cos y + \cos 2y\ dy, \\ &= \pi\left[4y - 4\sin y + \frac{1}{2}\sin 2y\right]_{0}^{\pi} = 4\pi^2. \end{aligned}$$

7. To find the volume we use cylindrical shells. The heights of the shells are given by $f(x) = 12(x^2 - x^3)$. Therefore, the volume is:

$$\begin{aligned} \text{volume} &= \int_{0}^{1} 2\pi x f(x)\ dx = \int_{0}^{1} 2\pi x[12(x^2 - x^3)]\ dx, \\ &= 24\pi\int_{0}^{1} x^3 - x^4\ dx = 24\pi\left[\frac{1}{4}x^4 - \frac{1}{5}x^5\right]_{0}^{1} = \frac{6\pi}{5}. \end{aligned}$$