RESEARCH STATEMENT

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My research interests are motivated by problems in quantum physics, especially problems in field and gauge theories. The analysis is usually done over an infinite dimensional space, for example a Hilbert space or a Hilbert manifold H. A path integral of the form

$$\frac{1}{Z}\int_{\sigma\in H} f(\sigma)e^{iS(\sigma)}\mathcal{D}_\sigma \tag{1}$$

is often used in physics literature. Here, f is some continuous function defined on H, S is typically the action of the underlying classical system and $\mathcal{D}$ is some type Lebesgue type of measure. Z is some normalization constant such that

$$\frac{1}{Z}\int_{\sigma\in H} e^{iS(\sigma)}\mathcal{D}_\sigma = 1.$$

Because there does not exist a Lebesgue measure on an infinite dimensional space, such an expression is only heuristic. To make sense of such an integral, one typically has to enlarge the underlying space H in order to define a probability measure on the enlarged space.

I am also interested in problems in stochastic analysis, for example, in the study of hypoelliptic operators on a manifold, heat kernel estimates and functional inequalities.

Chern-Simons Path Integrals

E. Witten in his famous paper in 1989 described how one can define a Jones polynomial of a link on an arbitrary manifold M. Start with a 3 manifold M. Let $E \to M$ be a trivial bundle over $\mathbb{R}^3$ and G a semisimple Lie group. Denote the Lie algebra of G by $\mathfrak{g}$. The Chern-Simons action is given by

$$\mathrm{CS}(A) = \frac{\kappa}{4\pi}\int_{\mathbb{R}^3} \mathrm{Tr}[A\wedge dA + \frac{2}{3}A\wedge A\wedge A]\ d\mathrm{vol}_{\mathbb{R}^3},\ \kappa\neq 0.$$

Here, A is some connection defined on $E \to M$. The goal is to make sense of Equation (1) with the Chern-Simons action. Such an integral will be referred as the Chern-Simons path integral.

Suppose we have a link $L \equiv \{C^k\}_{k=1}^r$ embedded inside M, each C^k is a closed curve. Let $\{\rho_k\}_{k=1}^r$ be any set of finite dimensional representations of G. For each curve C^k, one computes the holonomy of A along C^k,

$$W(C^k,\rho_k;q)(A) := \mathrm{Tr}_{\rho_k}\mathcal{T}\exp\left[q\int_{C^k} A_i dx^i\right].$$

Here, Tr_{ρ_k} is the matrix trace in the representation ρ_k and $\mathcal{T}$ is the time ordering operator.

The interest in Chern-Simons path integrals is the evaluation of

$$Z(M,C^i,\rho_i;q) := \frac{1}{Z_{\mathrm{CS}}}\int_{A\in\mathscr{A}}\prod_{k=1}^{l} W(C^k,\rho_k;q)e^{i\mathrm{CS}(A)}DA, \tag{2}$$

where DA is interpreted as some Lebesgue type of measure. $\mathscr{A}$ is the space of equivalent classes of connections, modulo gauge transformations. The integral in Equation (2) will be known as the Wilson Loop observable (associated to the link L). q will be called the charge of the link. E. Witten argued in his paper that the Wilson Loop observable can be used to define a Jones polynomial on M. However, his arguments are heuristic and thus making sense of Equation (2) rigorously is a challenge for many mathematicians.

In the case of $M = \mathbb{R}^3$, several authors were able to make sense of the Wilson Loop observable using white noise analysis. In the abelian gauge group G, it has been shown that it indeed defines a linking number of a link. For the non-abelian gauge group G, it still remains unresolved.

Instead of using white noise analysis, I used abstract wiener measure to make sense of the Chern-Simons path integral. One has to construct a probability space $\mathcal{B}$ and consider the space of $L^1(\mathcal{B})$. I will define the Chern-Simons path integral

$$f \in L^1(\mathcal{B}) \mapsto \int_{A \in \mathscr{A}} f(A) e^{i\mathrm{CS}(A)} DA,$$

as a linear functional defined on a dense subspace contained inside $L^1(\mathcal{B})$.

Using this definition, both the abelian and non-abelian cases are worked out explicitly. In the abelian case, the Chern-Simons path integral does indeed gives us the linking number. For the non-abelian gauge group, the Wilson Loop observable in Equation (2) can be written down as a state model Σ, defined using a R-matrix. To define a link invariant, a R-matrix has to satisfy certain algebraic equations, including the famous Yang-Baxter equation. Using this state model Σ, one is able to derive the correct skein relations that defines the Jones polynomial of a link embedded in $\mathbb{R}^3$.

Path Integrals on a manifold

It is standard folklore in the physics literature that a classical mechanical system may be "quantized" using Feynman path integrals. More explicitly, suppose that (M, g) is a Riemannian manifold, ∇ is the Levi-Civita covariant derivative, and $V : M \to \mathbb{R}$ is a potential. Then the informal path integral description of "the" quantum mechanical Hamiltonian associated to the classical mechanical system satisfying Newton's equations of motion,

$$\frac{\nabla^2}{dt^2} \sigma(t) = -\mathrm{grad}\, V(\sigma(t)),$$

is given by the heuristic expression

$$(e^{-T\hat{H}} f)(o) := \text{``}\frac{1}{Z_T} \int_{H_T(M)} e^{-\frac{1}{2}E_T(\sigma) - \int_0^T V(\sigma(r))\, dr} f(\sigma(T))\, \mathcal{D}\sigma.\text{''} \tag{3}$$

In this formula, $H_T(M)$ is the space of finite energy paths, $\sigma : [0, T] \to M$, with $\sigma(0) = o \in M$,

$$E_T(\sigma) := \int_0^T g(\sigma'(s), \sigma'(s))\, ds$$

is the energy of the path σ, Z_T is a certain normalization constant and $\mathcal{D}\sigma$ is some sort of " Lebesque " type measure.

My dissertation work involves approximating paths in $H_T(M)$ by piecewise geodesic paths and then passing to the limit to make rigorous sense of the right side of Equation (3). Let $\mathcal{P}$ be a partition of $[0, T]$ and define $H_{\mathcal{P},T}(M) \subseteq H_T(M)$ to be the space of piecewise geodesic paths, changing directions only at the partition points. It can be shown to be diffeomorphic to a finite dimensional vector space. Using a H^1 metric defined on the Hilbert manifold $H_T(M)$, its restriction defines a Riemannian metric and a volume form $\nu_{\mathcal{P},T}$ on $H_{\mathcal{P},T}(M)$. Let $W_T(M)$ be the space of continuous paths, $\sigma : [0, T] \to M$, with $\sigma(0) = o$ equipped with Wiener measure ν_T on $W_T(M)$. Suppose that M is a $d-$ dimensional compact manifold Riemannian manifold with positive sectional curvature. For any continuous function f on $W_T(M)$, I showed that by using a suitable normalization constant $Z_{\mathcal{P},T}$, as the partition size $|\mathcal{P}|$ tends to 0,

$$\lim_{|\mathcal{P}| \to 0} \frac{1}{Z_{\mathcal{P},T}} \int_{\sigma H_{\mathcal{P},T}(M)} f(\sigma) e^{-E_T(\sigma)} d\nu_{\mathcal{P},T} = \int_{W_T(M)} f(\sigma(T)) \rho_T(\sigma)\, d\nu_T(\sigma). \tag{4}$$

Here, ρ_T is some density function defined on $W_T(M)$, which can be written down explicitly in terms of the curvature of the underlying manifold M. In the flat case, ρ_T is exactly equal to 1.

Suppose one defines an operator $\hat{H} : f \in C^\infty(M) \mapsto \hat{H} f \in C^\infty(M)$ using the RHS of Equation (4),

$$e^{-T\hat{H}} f(o) := \int_{W_T(M)} f(\sigma(T)) \rho_T(\sigma)\, d\nu_T(\sigma).$$

$\hat{H}$ can be recovered by differentiating the RHS of Equation (4) with respect to time T. A subsequent work of mine showed that $\hat{H}$ is a second order elliptic operator, plus a first order derivative term and a multiplication operator by scalar curvature.

Analysis on free Loop space

Suppose we have a spin manifold (M, g). Now, consider $L(M)$, the space of free (smooth) loops in M. It is known that $L(M)$ is spin if the first pontryagin class vanishes. If $L(M)$ is spin, then one can construct a spin bundle S over $L(M)$.

Let (M, g) be orientable. Suppose $\sigma \in L(M) : [0, 1] \rightarrow M$. Equip $L(M)$ with a metric, $G : X, Y \in T_\sigma L(M) \mapsto \int_0^1 g(X, Y)(\theta)d\theta$. I worked out the Levi-Civita covariant derivative and its curvature. Furthermore, I have constructed a probability measure on free loop space.

Future work aims to construct a Dirac type of operator on $L(M)$. Such an operator can be used to compute certain topological invariants on the spin manifold M. It is well known that if scalar curvature is positive, then the total $\hat{A}$ class of M vanishes. Stolz conjectured that if M has positive Ricci, then the Witten genus of M vanishes. A heuristic argument given by Stolz uses a (non-existent) Dirac operator constructed on the spin bundle S over $L(M)$ to prove the conjecture.

Current and Future Research

1. The Chern-Simons path integral was worked out for $\mathbb{R}^3$. Unfortunately little progress is done on other type of manifolds. Now, I am working on extending the results to $M \times S^1$, whereby M is a 2 manifold. Some of the computations of the Wilson loops observables can be applied to other areas, for example to problems in quantum gravity.

2. The construction of the probability measure in the Chern-Simons path integral can be applied to quantum field theory. The Wilson Loop observable in $\mathbb{R}^3$ can be computed directly from the link diagram. This is similar to the Feynman diagrams. It is hoped that by using the abstract wiener measure formulation, one can define rigorously the path integrals used to compute the Green's functions in quantum field theory. More importantly, I hope to be able to explain rigorously how one can use Feynman diagrams to compute these path integrals.

3. Elliptic operators are an important class of unbounded linear operators and tools in stochastic analysis have obtained many functional inequalities for these operators, for example, the log Sobelov inequality, Poincare Inequality and the Harnack Inequality with Power. Unfortunately, many of these methods fail in the case of hypoelliptic operators. However, certain functional inequalities do hold for certain hypoelliptic operators. I have been working on extending the Harnack Inequality with Power that holds for elliptic operators to that of hypoelliptic operators.

4. Start with a manifold M^n. Given any positive solution to the heat equation $\Delta u = \partial_t u$, define an entropy $Ent_{M^n}(u) = \int_M u \log u d\mu$, μ is the volume form on M. It was shown by L. Ni that if Φ_n is the fundamental solution to the heat equation on $\mathbb{R}^n$, then

$$\partial_t(Ent_{M^n}(u) - Ent_{\mathbb{R}^n}(\Phi_n)) < 0,$$

if M^n has positive Ricci. In a recent preprint of mine, I have shown that the result can be extended; if Φ_n is the fundamental solution to the heat equation on the sphere S^n, then

$$\partial_t(Ent_{M^n}(u) - Ent_{S^n}(\Phi_n)) < 0.$$

Current work hopes to obtain similar results for manifolds with negative curvature.