Vacant set of random walk on (random) graphs.

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joint work with
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General Problem

Model:

- $G = (V, \mathcal{E})$ – a large finite connected graph, (constant degree)
- $(X_t)_{t \geq 0}$ – simple random walk on G, started from its invariant (uniform) distribution.
- $\mathcal{V}^u = V \setminus \{X_t : t \leq u|V|\}$ – **vacant set** at time $u|V|$. $u > 0$ – parameter

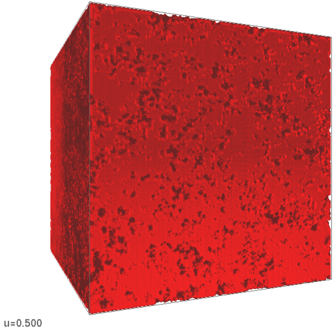
Question: Structure (percolative properties) of the vacant set $\mathcal{V}^u$.

Remark. Scaling $u|V|$ in the definition of $\mathcal{V}^u$:

$$P[x \in \mathcal{V}^u] \sim \rho(u) \in (0, 1), \qquad x \in V.$$

Example: d-dimensional torus $(\mathbb{Z}/n\mathbb{Z})^d$

$V = (\mathbb{Z}/n\mathbb{Z})^d$, $n \in \mathbb{N}$, $d \geq 3$, nearest neighbour edges.



Simulation by D. Windisch

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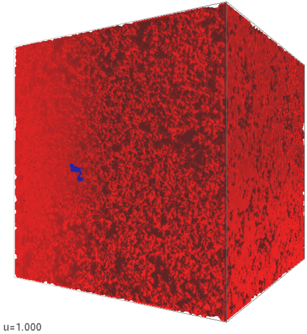

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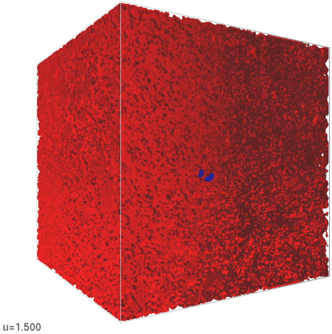

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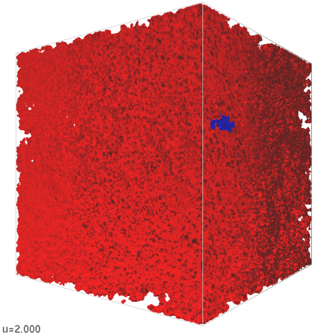

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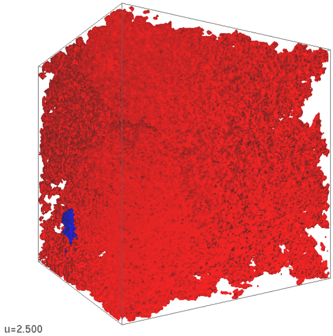

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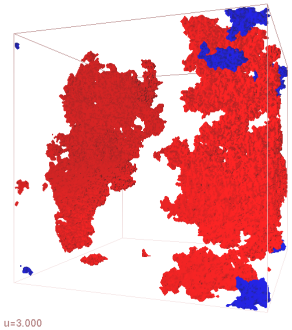

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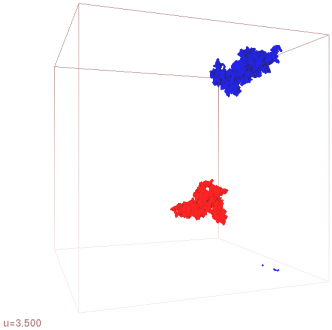

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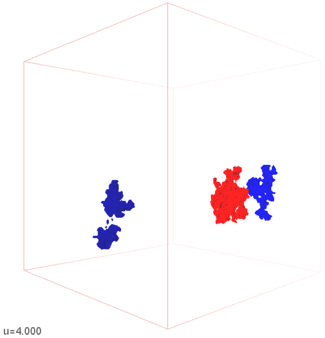

Simulation by D. Windisch

Phase transition?!

Problem I: Existence of a phase transition

- $G_n = (V_n, \mathcal{E}_n)$ – sequence of finite connected graphs converging to a **transient** infinite graph $\mathbb{G} = (\mathbb{V}, \mathcal{E}_{\mathbb{G}})$.

  ($\exists r_n \to \infty$, s.t. for a typical $x \in V_n$: $B_{G_n}(x, r_n) \overset{\phi_n^x}{\simeq} B_{\mathbb{G}}(0, r_n)$).

- $\mathcal{V}_n^u = V_n \setminus \{X_t^n : t \in [0, u|V_n|]\}$ – **vacant set**,

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Is there a phase transition?

Is there $u_c = u_c(G_n) \in (0, \infty)$ such that

- *Supercritical phase.* For $u < u_c$ there is a giant component:

$$\exists c(u) > 0 \text{ such that } P[|\mathcal{C}_{\max}^{u,n}| \geq c|V_n|] \xrightarrow{n\to\infty} 1.$$

- *Subcritical phase.* For $u > u_c$ all components are small:

$$P[|\mathcal{C}_{\max}^{u,n}| \ll n] \xrightarrow{n\to\infty} 1.$$

Prior results

For the d-**dimensional torus**:

- Benjamini-Sznitman (JEMS '08):
  If u is small enough, then $\mathcal{V}^u$ has a giant component.
- improved slightly by D. Windisch (EJP '08)
- recent considerable improvements by [WT10]

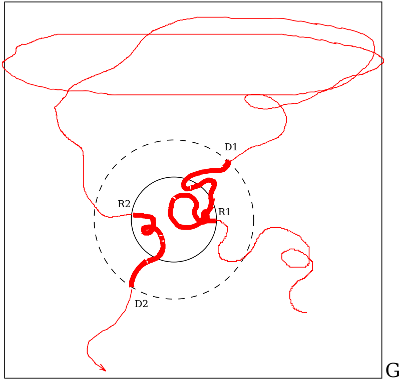
Related literature:

- disconnection of discrete cylinder $G_n \times \mathbb{Z}$
  – Dembo, Sznitman; Sznitman 2006–2009
- Random interlacement

Random interlacement - motivation

Percolation model on an infinite graph $\mathbb{G} = (\mathbb{V}, \mathcal{E}_{\mathbb{G}})$

Question. A local limit for the vacant set

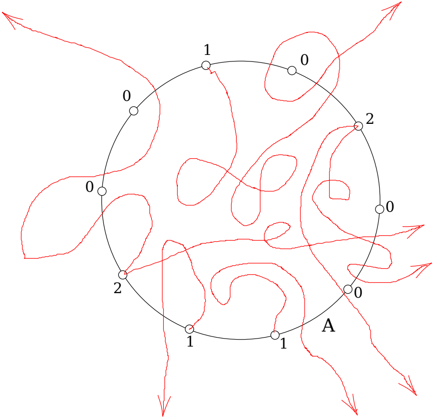


Visits of a ball in the *finite* graph

Random interlacement - definition

Local construction of RI: Let $A \subset \mathbb{V}$ finite.

- *equilibrium measure:*
  $e_A(x) = \mathrm{Prob}[\text{RW on } \mathbb{V} \text{ started at } x \text{ never returns to } A] \cdot \mathbf{1}_A(x)$.
- at every point x start Poisson($ue_A(x)$) independent random walks
- vacant set of RI:
  $\mathcal{V}^u_{\mathsf{RI}}|_A$ = the set of vertices in A *not visited* by these random walks.



Random interlacement - definition

The extension to the whole $\mathbb{G}$:

$$\text{If } A \subset B \text{ finite , then } (\mathcal{V}_{\mathsf{RI}}|_B)|_A \stackrel{\text{law}}{=} \mathcal{V}_{\mathsf{RI}}|_A.$$

Constructed on $\mathbb{Z}^d$ in Sznitman, Ann. Math 2010,
Extended to transient graphs in Teixeira, EJP 2009.

Random interlacement - phase transition

Critical point of RI: $u_\star(\mathbb{G})$,

- If $u < u_\star$, then $\mathcal{V}^u_{\mathsf{RI}}$ contains an infinite connected component $\mathbb{P}$-a.s.
- If $u > u_\star$, then there are $\mathbb{P}$-a.s. only finite components of $\mathcal{V}^u_{\mathsf{RI}}$.

Theorem. (Sznitman, Sidoravicius) $u_\star$ exists and is non-trivial:

$$0 < u_\star(\mathbb{Z}^d) < \infty \qquad \text{for all } d \geq 3.$$

Problem II: Relation of two models

Theorem. (D. Windisch, ECP 2008)
$\mathcal{V}^u_{\mathsf{RI}}$ is a **local limit** of the vacant set $\mathcal{V}^u$ on the torus $(\mathbb{Z}/n\mathbb{Z})^d$.

Remark. Results on Random Interlacement can be used to prove:

- For $u < u_{\star\star\star} \overset{?}{\leq} u_\star(\mathbb{Z}^d)$, there is a giant component
- For $u > u_{\star\star} \overset{?}{\geq} u_\star(\mathbb{Z}^d)$, the largest component has size $O(\log^K n)$
- For $u > u_\star(\mathbb{Z}^d)$, the largest component has size $o(n)$, [WT10]

Conjecture.

$$u_c(G_{n}) = u_\star(\mathbb{G}).$$

Problem II. Prove this conjecture.

Our setting

Consider graphs that are simpler for Bernoulli percolation:

- d**-regular large-girth expanders**
  like Ramanujan or Lubotzky-Phillips-Sarnak graphs
- **random** d**-regular graph**
  (graph uniformly chosen from all d-regular graphs on n vertices)

Both these classes of graphs are *"finite approximations of d-regular tree"*

Bernoulli percolation on such graphs studied by Alon-Benjamini-Stacey '04, Nachmias-Peres '09, Pittel '09.

A sequence G_n is expander if for some $c > 0$

$$\frac{|\partial A|}{|A|} \geq c, \qquad \forall n, \ \forall A \subset V_n, |A| < |V_n|/2.$$

Our setting

Assume that G_n satisfies:

(A0) $G_n = (V_n, \mathcal{E}_n)$ is d-regular, $|V_n| = n$.

(A1) *Local almost tree-like property:*
There exists $\alpha_1 \in (0,1)$ such that for all n and $x \in V_n$

the ball $B(x, \alpha_1 \log n)$ contains at most one cycle

(A2) *Uniform spectral gap:*
There exists $\alpha_2 > 0$ such that for all n: $\lambda_1(G_n) \geq \alpha_2$

Remarks

- random d-regular graph satisfies (A0)–(A2) **whp**.
- (A1): typical $x \in V_n$ has *tree-like* neighbourhood.
- (A2) is equivalent (via Cheeger's inequality) to expansion

Results: Phase transition

Theorem. Let G_n satisfy (A0)–(A2). Then there exists $u_c(d)$

1. **(giant component)** For $u < u_c$ exists $\rho > 0$ such that

$$|\mathcal{C}_{\max}(\mathcal{V}^u)| \geq \rho n \qquad \textbf{whp}$$

2. **(uniqueness)** For $u < u_c$, for every $\varepsilon > 0$,

$$|\mathcal{C}_{\text{sec}}(\mathcal{V}^u)| \leq \varepsilon n \qquad \textbf{whp}$$

3. **(subcritical phase)** For $u > u_c$, there is $K < \infty$

$$|\mathcal{C}_{\max}(\mathcal{V}^u)| \leq K \log n \qquad \textbf{whp}$$

Results: Relation to Random Interlacement

Theorem. (equality of critical points)
Let G_n satisfy (A0)–(A2) and let $\boldsymbol{T}_d$ be the d-regular tree

$$u_c(d) = u_\star(\boldsymbol{T}_d).$$

Problem III: Critical behaviour

Question. Behaviour of the model when $u = u_c(d)$ or $u_n \to u_c(d)$.

In the Bernoulli percolation there is the Erdős-Rényi double jump:

- When $|p_n - p_c| \leq cn^{-1/3}$, then $|\mathcal{C}_{\max}| \sim n^{2/3}$.
- When $p_n - p_c \to 0$ and $n^{1/3}(p_n - p_c) \to \infty$, then $|\mathcal{C}_{\max}| \gg n^{2/3}$.
- When $p_n - p_c \to 0$ and $n^{1/3}(p_n - p_c) \to -\infty$, then $|\mathcal{C}_{\max}| \ll n^{2/3}$.

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Does the vacant set of the random walk exhibit a similar behaviour?

Results: Critical behaviour of the vacant set

We can consider random d-regular graphs only!

Define

- $\mathbb{P}_{n,d}$, the distribution of the random d-regular graph G on n vertices
- P^G, the distribution of the RW on the graph G
- $\mathbf{P}_{n,d}$, the averaged distribution of the RW,

$$\mathbf{P}_{n,d}(\cdot) = \int P^G(\cdot)\mathbb{P}_{n,d}(\mathrm{d}G).$$

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Theorem. [ČT'2011]

- When $|n^{1/3}(u_n - u_\star)| \le \lambda < \infty$, then $\forall \varepsilon > 0 \exists A$ s.t. $\forall n > n_0$

$$\mathbf{P}_{n,d}[A^{-1}n^{2/3} \le |\mathcal{C}^{u_n}_{\max}| \le An^{2/3}] \ge 1 - \varepsilon.$$

- When $u_\star - u_n \to 0$ and $\omega_n := n^{1/3}(u_\star - u_n) \to \infty$, then

$$|\mathcal{C}^{u_n}_{\max}| \sim c(d)\omega_n n^{2/3}, \qquad \mathbf{P}_{n,d}\text{-a.a.s.}$$

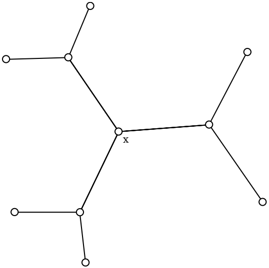
- When $u_\star - u_n \to 0$ and $\omega_n \to -\infty$,

$$|\mathcal{C}^{u_n}_{\max}| \le Bn^{2/3}|\omega_n|^{-1/2} \qquad \mathbf{P}_{n,d}\text{-a.a.s.}$$

Random Interlacement on tree $\boldsymbol{T}_d$

Lemma. (Teixeira, 2009) Given $x \in \mathcal{V}_{\mathsf{RI}}$, the cluster $\mathbb{C}_x$ of x has the law of branching process whose offspring distribution is binomial with parameters $d-1$ (resp. d in the first generation) and p_u, where

$$p_u = \exp\Big\{ -\frac{u(d-2)^2}{d(d-1)} \Big\}.$$



Consequence. $u_\star(\boldsymbol{T}_d)$ is the solution to $(d-1)p_{u_\star} = 1$.

Local convergence to Random Interlacement

Lemma. There is $\beta \in (0, \alpha_1/5)$, such that for all x with tree-like neighbourhood of radius $r := 5\beta \log_{d-1} n$, for all $u > 0$, $\varepsilon > 0$, there exists a coupling $\mathbb{P}$ of RW on G and RI's on $\boldsymbol{T}^d$ such that

$$\mathbb{C}_0^{u-\varepsilon}\Big|_{B_{\mathbb{G}(0,r)}} \overset{\phi_n^x}{\supset} \mathcal{C}_x(\mathcal{V}^u)\Big|_{B_{G_n}(x,r)} \overset{\phi_n^x}{\supset} \mathbb{C}_0^{u+\varepsilon}\Big|_{B_{\mathbb{G}}(0,r)} \qquad \textbf{whp}(\mathbb{P}).$$

Consequence. In every tree like ball of radius $\beta \log_{d-1} n$ we have a good control of $\mathcal{C}_x(\mathcal{V}^u)$ by a branching process.

But we need more!

THE LOCAL CONTROL IS NOT SUFFICIENT!

- In the super-critical phase, the giant component cannot be contained in a ball of of radius $\beta \log_{d-1} n$ (and thus volume $< n^{\beta}$, $\beta < 1$).
- In the sub-critical phase, the largest cluster has diameter $\sim K \log_{d-1} n$, but $K \to \infty$ as $u \downarrow u_c$.

  In particular, since $\operatorname{diam} G = \log_{d-1} n(1 + o(1))$, we have

$$\operatorname{diam} \mathcal{C}_{\max}(\mathcal{V}^u) \geq \operatorname{diam} G.$$

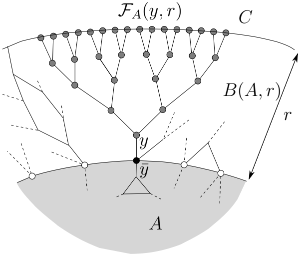
Proof Ideas: Sub-critical regime

Localisation. Prove that $\forall x$

$$P[\mathcal{C}_x \geq K \log n] \leq \varepsilon / n.$$

Stochastic breadth-first-search algorithm.

- Construct $\mathcal{C}_x$ step by step by a BFS exploration.
- Control the probability that the next vertex is added to $\mathcal{C}_x$. This can be done only in a specific situation ($r = 7 \ln \ln n$):



- Goal: algorithm stops before $(K \log n)$-th steps.

Proof Ideas: Super-critical regime

Sprinkling.

1. Fix $u' \in (u, u_c)$.
2. Consider $\mathcal{V}^{u'}$ and use the local branching-process comparison to construct many large components:

$$\#\{x : |\mathcal{C}_x(\mathcal{V}^{u'})| \geq n^\beta\} \geq \rho n \qquad \textbf{whp}.$$

3. Erase 'some $(u' - u)n$ points' of the trajectory'
   Problem. Cannot erase the last part of the trajectory.
4. After the erasure, many of the components constructed in point 2 merge to a unique giant component.

Proof Ideas: Critical window

- In the random regular graph case, the vacant set is distributed as random graph with a given (random) degree sequence
  - Cooper-Frieze 2010.

  Lemma. Let
  - d_x^u be the degree of $x \in V_n$ in the subgraph of G generated by $\mathcal{V}^u$,
  - Q_n^u be the distribution of $\boldsymbol{d} = (d_x^u)_{x \in V_n}$ under $\boldsymbol{P}_{n,d}$.
  - $\mathbb{P}_{\boldsymbol{d}}$ the distribution of the uniformly chosen graph with degree sequence $\boldsymbol{d}$.

  Then

$$\boldsymbol{P}_{n,d}(\mathcal{V}^u \in \cdot) = \int \mathbb{P}_{\boldsymbol{d}}[G \in \cdot] Q_n^u(\mathrm{d}\boldsymbol{d})$$

Proof Ideas: Critical window

- Random graphs with a given degree sequence are well understood:
  Phase transition. Molloy-Reed 1993,
  Critical regime. Hatami-Molloy 2010.

  Theorem. Let $\boldsymbol{d}^n = (d_1^n, \ldots, d_n^n)$ be a deterministic sequence of degree sequence. Set

  $$\mathcal{Q}(\boldsymbol{d}) = \frac{\sum_x d_x^2}{\sum_x d_x} - 2.$$

  Then $\mathbb{P}_{\boldsymbol{d}^n}$-a.a.s.

  - $\lim_{n\to\infty} \mathcal{Q}(\boldsymbol{d}^n) > 0 \implies$ giant component
  - $\lim_{n\to\infty} \mathcal{Q}(\boldsymbol{d}^n) < 0 \implies$ only log-size components
  - $\mathcal{Q}(\boldsymbol{d}^n) \sim n^{-1/3} \implies$ critical window.

- To prove our result, we need only to control the distribution Q_n^u of the degree sequence of the vacant set with sufficient precision.

Open problems

1. Density of giant cluster for $u < u_c$.
2. Stronger uniqueness result. Is $|\mathcal{C}_{\text{sec}}| = O(\log n)$.
3. What about $u = u_c$, out of the random d-regular graph case?

4. Other graphs, TORUS?

The end

Thank you for your attention.