Transition phase for the speed of the biased random walk on a percolation cluster

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The biased random walk on a percolation cluster

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The mathematical model

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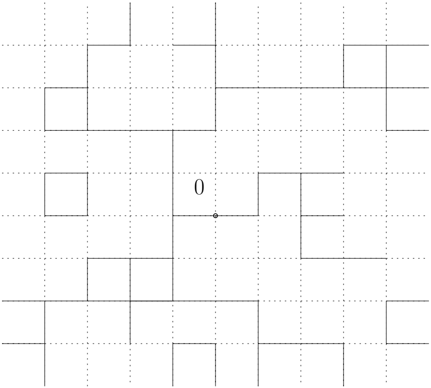
Percolation

Let us consider $\mathbb{Z}^d$ for $d \geq 2$. We perform an edge percolation of parameter p

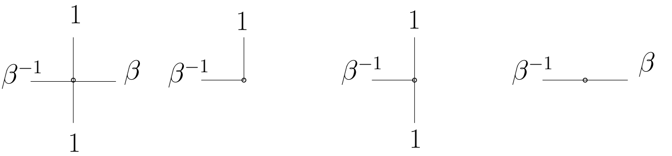
- Each edge of $\mathbb{Z}^d$ is open with probability p independently of all others.



Biased random walk

For simplicity in this talk assume the drift is along a direction e_1 of the grid. We take $\exp(\lambda)$ with $\lambda > 0$ to be its strength.

In the environment ω the transitions probabilities are given by



with $\beta = \exp(\lambda)$ $\longrightarrow$ direction of the drift

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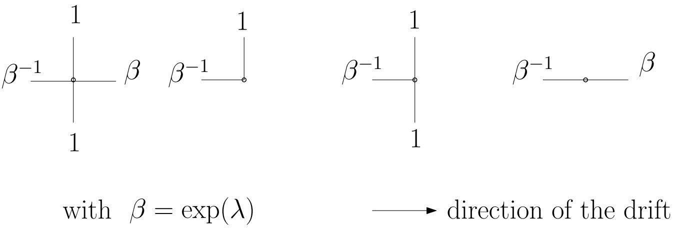

We can define a similar model with a drift in any direction.

Reasons for studying this model

This model was first considered in the physics literature (Barma, Dhar (83); Dhar (84); Dhar, Stauffer (98)).

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It is a challenging problem of RWRE, considered by Berger, Gantert, Peres (03); Sznitman (03). It is not uniformly elliptic and the environment is asymmetric.

$$c(x,y) = \exp(\lambda(x+y)\cdot e_1)\mathbf{1}\{[x,y]\text{open}\}.$$

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Representative of RWRE with directional transience and trapping.

Results

Berger, Gantert, Peres and Sznitman 2003

Fix p and d. The random walk is transient in the direction e_1, i.e.

$$\lim X_n \cdot e_1 = \infty, \quad P^\omega\text{-a.s. for } \omega - \mathbf{P}_p\text{-a.s.}.$$

Moreover

$$\lim \frac{X_n}{n} = v(\lambda), \quad P^\omega\text{-a.s. for } \omega - \mathbf{P}_p\text{-a.s.}.$$

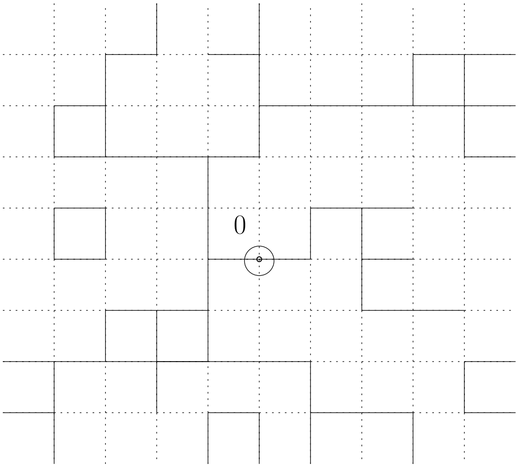
There exists $\lambda_c^{(1)} \geq \lambda_c^{(2)} > 0$ such that

- if $\lambda < \lambda_c^{(2)}$, then $v(\lambda) \cdot e_1 > 0$,
- if $\lambda > \lambda_c^{(1)}$, then $v(\lambda) = 0$.

Behavior of the walk

Behavior of the walk

Local behavior



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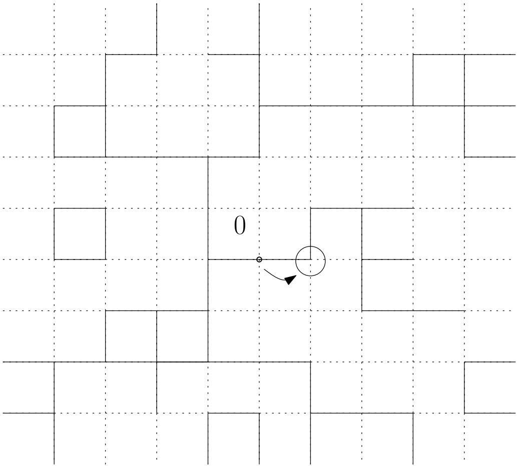

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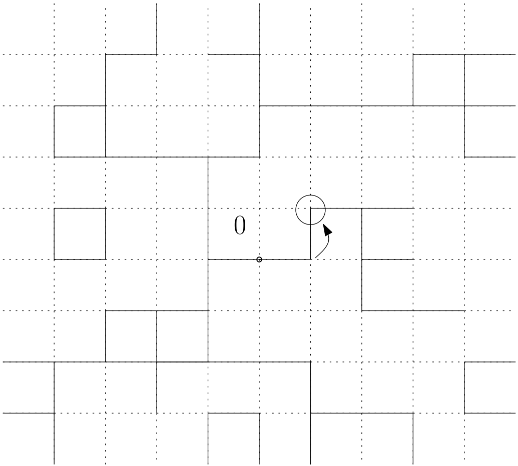

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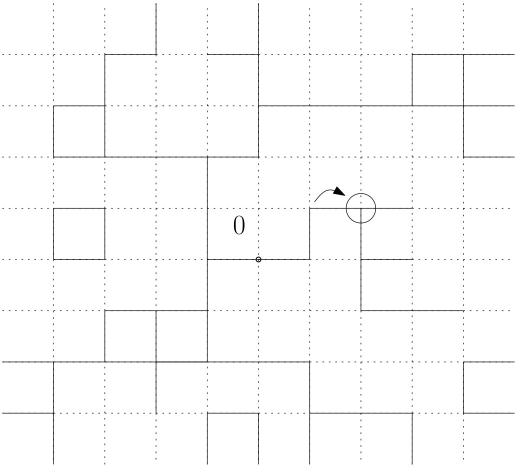

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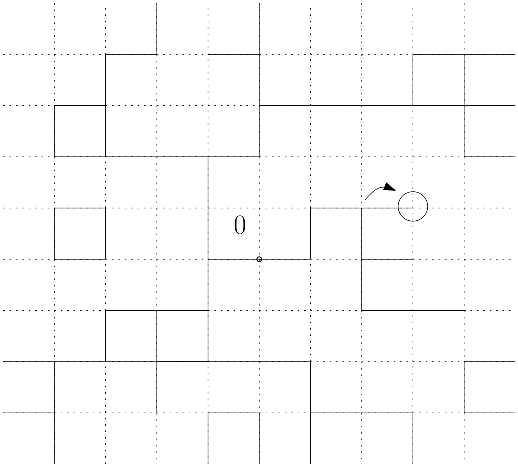

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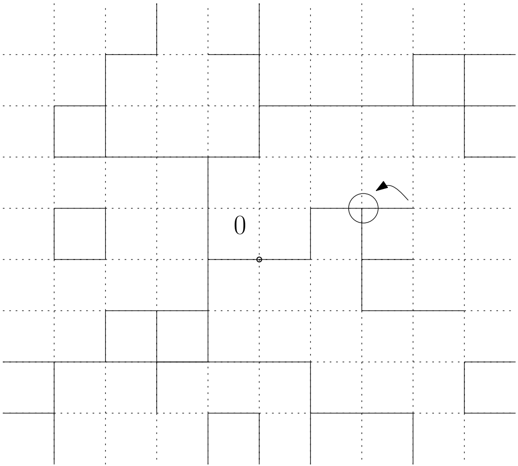

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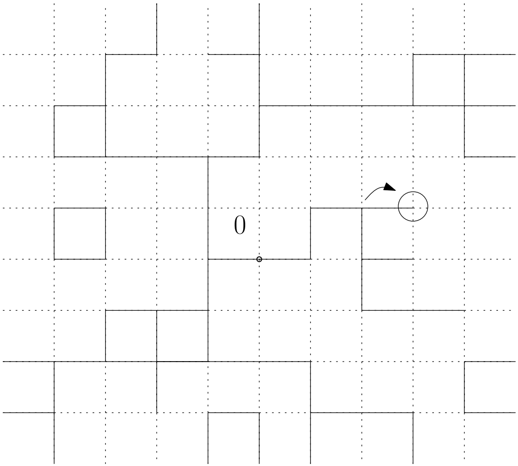

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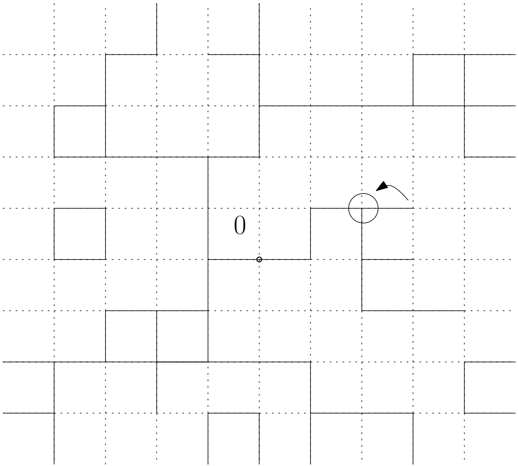

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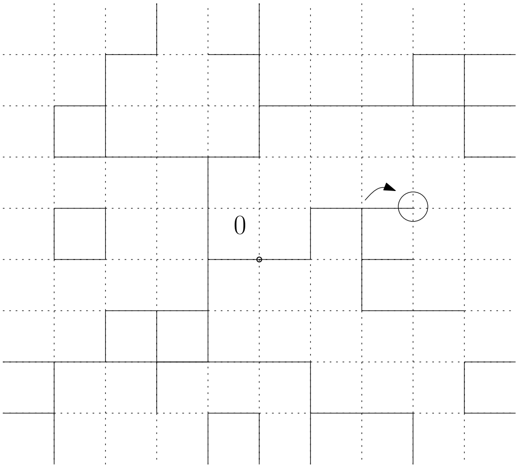

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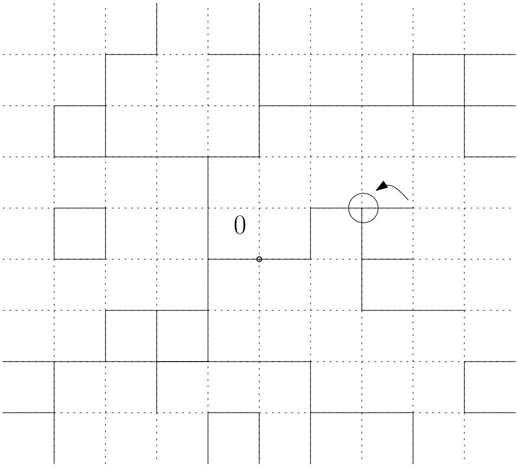


Local behavior

There are traps in the environment.

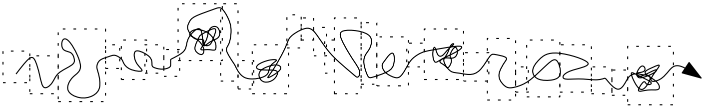
To exit a trap the walk has to backtrack (go opposite to the drift) for n steps. It takes

$$T_{\text{exit}} \approx \exp(2\lambda n),$$

units of time to do so.

Global behavior

The trajectory looks uni-dimensional.



The phase transition

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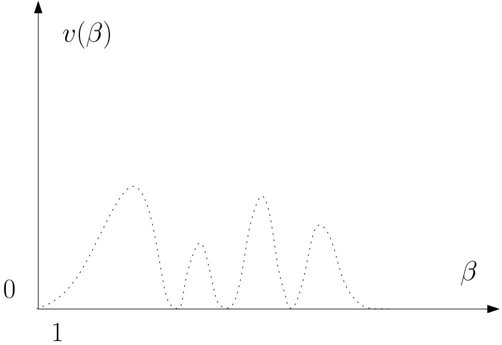
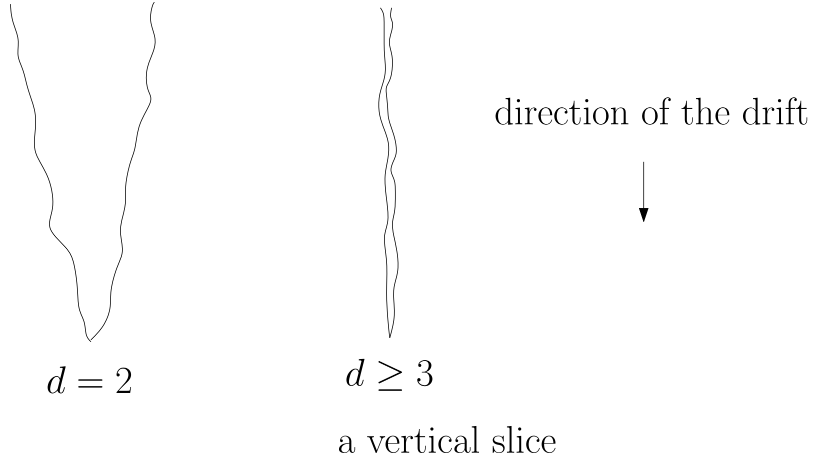
The issue



One important conjecture is that $\lambda_c^{(1)} = \lambda_c^{(2)}$ (phase transition).

Backtracking function

The most effective type of traps we encounter (represented dually)



Backtracking function

Let us introduce the number of steps you need to backtrack to exit the trap at 0

$$\mathcal{BK} = \min_{(p(i))_{i\in\mathbb{N}}\in\mathcal{P}} \max_{i\geq 0} p(i)\cdot(-e_1),$$

where $\mathcal{P}$ is the set of infinite open self-avoiding paths starting from 0.
We can prove that there exists $\xi(p,d)\in(0,\infty)$ such that

$$\mathbf{P}[\mathcal{BK}\geq n \mid 0 \text{ is in the infinite cluster}] \approx \exp(-\xi(p,d)n).$$

Transition phase for the speed

F. and Hammond 2011

In $\mathbb{Z}^d$, let us introduce $\gamma = \frac{\xi}{2\lambda}$. We have

$$\lim \frac{X_n}{n} = v, \qquad P^\omega\text{-a.s. for } \omega - \mathbf{P}_p\text{-a.s.},$$

where

$$\text{if } \gamma > 1, \text{ i.e. } \lambda < \xi/2, \text{ then } v \cdot e_1 > 0,$$

$$\text{if } \gamma < 1, \text{ i.e. } \lambda > \xi/2, \text{ then } v = 0.$$

Moreover, if $\gamma \leq 1$ then

$$\lim \frac{\ln X_n \cdot e_1}{\ln n} = \gamma, \qquad P^\omega\text{-a.s. for } \omega - \mathbf{P}_p\text{-a.s.}.$$

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Sketch of proof

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Hitting time of level n

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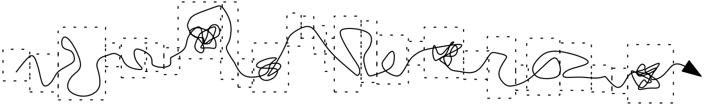

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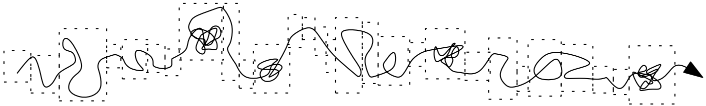

we see that ideally the hitting time of the level n is essentially the time spent in the Cn first traps encountered.

Hence the system is similar to a one-dimensional Bouchaud Trap Model (as conjectured by Sznitman 06).

Hitting time of level n

Thus the hitting time of the level n is an i.i.d. sum

$$\Delta_n \approx \sum_{i=0}^{Cn} T_{exit}^{(i)},$$

is a sum of i.i.d. times to exit traps. Hence we need to know if the expectation of $T_{exit}^{(i)}$ is finite in an averaged sense.

Time spent in a trap

We recall that

1. We need $\exp(2\lambda n)$ units of time to backtrack n steps in a trap,
2. the number of steps we backtrack is typically $\mathcal{BK}$, which is such that

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From this we can see that averaged over the environment

$$\mathbb{P}\left[T_{exit}^{(i)} \geq t\right] = \mathbb{P}\left[\exp(2\lambda\mathcal{BK}) \geq t\right] \approx \mathbb{P}\left[\mathcal{BK} \geq \frac{1}{2\lambda}\ln t\right] \approx t^{-\gamma},$$

with $\gamma = \xi/2\lambda$.

Time spent in a trap

In the end

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Then

1. if $\gamma > 1$, $\mathbb{E}[T_{exit}^{(i)}] < \infty$ so $\Delta_n \approx Cn$ and $v = X_n/n \approx 1/C$,
2. if $\gamma < 1$, $\mathbb{E}[T_{exit}^{(i)}] = \infty$ so $\Delta_n \approx \infty n$ and $v \approx 1/\infty = 0$.

The key ingredients

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Technical aspects

1) Condition implying $(T)_\gamma$

Lemma

Denoting $B(L, L^\alpha) = [-L, L]_{e_1} \times [-L^\alpha, L^\alpha]_{e_1^\perp}^{d-1}$. *For large* α

$$\mathbb{P}[X_n \text{ does not exit } B(L, L^\alpha) \text{ in the positive direction}] \le ce^{-cL}.$$

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$$\mathbb{P}[X_n \text{ does not exit } B(L, L^\alpha) \text{ in the positive direction}] \leq ce^{-cL}.$$

This lemma proves that regeneration boxes are small ($\approx \ln^C t$).
So the walk is truly one-dimensional.

Technical aspects

2) Exit time of a box

Lemma

For any α, we have for all L not too big (e.g. $\approx \ln^C t$)

$$\mathbb{P}[T^{exit}_{B(L,L^\alpha)} \geq t] \approx t^{-\gamma}.$$

Technical aspects

2) Exit time of a box

Lemma

For any α, we have for all L not too big (e.g. $\approx \ln^C t$)

$$\mathbb{P}[T^{exit}_{B(L,L^\alpha)} \geq t] \approx t^{-\gamma}.$$

This lemma reflects that the time spent in a small box (e.g. a regeneration box) is mainly given by the time spent in traps.

Proof of $(T)_\gamma$

1. Exit times for biased reversible random walks can be efficiently approximated through spectral estimates (by Saloff-Coste).

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   tells us that in short time we can only be at places not too far from 0 and with high invariant measure
3. By replacing the original graph by one where the traps have been replaced by edges, we can
   - conserve exit probabilities
   - speed up the walk

Related models

Related models

Another model of biased random walks

Consider the biased random walk in positive conductances c_* (Shen 02).

F. 2011

For $d \geq 2$, we have

$$\lim \frac{X_n}{n} = v, \qquad \mathbb{P} - \text{a.s.},$$

where

1. if $E_*[c_*] < \infty$, then $v \cdot \vec{\ell} > 0$,
2. if $E_*[c_*] = \infty$, then $v = 0$.

Open problems

Open problems

1. understand the scaling limits,
2. Einstein relation,
3. understand the behavior of the speed in the ballistic regime.

Thanks!

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Tail estimates on regeneration times

The idea is that if the time spent in a regeneration box is large then

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2. or the walk spends a lot of time in a small box.

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1. either the regeneration box is large ($\geq \ln^C t$),
2. or the walk spends a lot of time in a small box.

The first part is a consequence of $(T)_\gamma$.

The second one has probability $t^{-\gamma}$ from our estimates on exit time of boxes.

Small regeneration boxes

Condition $(T)_\gamma$ implies little backtracking

$$\mathbb{P}[(X_{\tau_2} - X_{\tau_1}) \cdot \vec{\ell} \geq t] \leq Ce^{-ct^{1/3d}},$$

so we have small variations along $\vec{\ell}$.

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Condition $(T)_\gamma$ then also implies little variations in orthogonal directions. So introducing the volume of a regeneration box

$$\mathrm{Vol}_\tau = \inf\{k,\ (X_i - X_{\tau_1})_{i\in[\tau_1,\tau_2]} \subset B(k, k^\alpha)\},$$

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$$\mathrm{Vol}_\tau = \inf\{k,\ (X_i - X_{\tau_1})_{i \in [\tau_1, \tau_2]} \subset B(k, k^\alpha)\},$$

we have tails on the size of boxes.

$$\mathbb{P}[\mathrm{Vol}_\tau \geq k] \leq ce^{-ck^{1/3d}} \text{ or } \mathrm{Vol}_\tau \leq \ln^C t, \text{ w.h.p.}$$