Crossing velocities for annealed random walks in a random potential

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Outline

Random walk in a random potential (RWRP)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(V(x,\omega))_{x\in\mathbb{Z}^d}$ be i.i.d. under $\mathbb{P}$, and $V(0) \in [0,\infty]$, with $\mathbb{P}(V(0) < \infty) > 0$ and $\mathbb{P}(V(0) = 0) < 1$. A good example to have in mind is when $V(0) \in \{0, 1, \infty\}$ with $\mathbb{P}(V(0) = 0) \in (0,1)$. For $\omega \in \Omega$ consider a "killed" random walk $(S_j)_{j\geq 0}$ on $\mathbb{Z}^d \cup \{\dagger\}$, where $\dagger$ is an absorbing state, and for $x \in \mathbb{Z}^d$

$$p(x,x') = \begin{cases} \dfrac{1}{2d}\, e^{-V(x,\omega)}, & \text{if } \|x' - x\| = 1; \\ 1 - e^{-V(x,\omega)}, & \text{if } x' = \dagger; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that if $V(x,\omega) = \infty$ then the walk is "killed" upon reaching x with probability 1. In this case we say that there is a "hard obstacle" at x. Those x for which $V(x,\omega) \in (0,\infty)$ correspond to "soft obstacles". Sites where $V(x,\omega) = 0$ will be called "empty".

RWRP conditioned to hit a remote location

Let $S_0 = 0$. For $y \in \mathbb{Z}^d$ set $\tau_y = \inf\{j \geq 0 : \; S_j = y\}$. We shall be interested in $\|y\| \gg 1$.
For $\omega \in \Omega$ define

$$Z_y^\omega = \int e^{-\sum_{j=0}^{\tau_y - 1} V(S_j, \omega)} 1_{\{\tau_y < \infty\}} \, dP^0$$

be the probability that the "killed" random walk reaches y in potential $V(\cdot, \omega)$. Here P^0 is the probability measure of the simple symmetric random walk that starts from 0. When $Z_y^\omega \neq 0$ we can define the ***quenched path measure***

$$Q_y^\omega(A) = \frac{1}{Z_y^\omega} \int_A e^{-\sum_{j=0}^{\tau_y - 1} V(S_j, \omega)} 1_{\{\tau_y < \infty\}} \, dP^0.$$

RWRP conditioned to hit a remote location

Let $S_0 = 0$. For $y \in \mathbb{Z}^d$ set $\tau_y = \inf\{j \geq 0 : \ S_j = y\}$. We shall be interested in $\|y\| \gg 1$.
Let $Z_y = \mathbb{E} Z_y^\omega$. Then $Z_y > 0$ and we can define the ***annealed measure*** (on the product space of paths and environments)

$$Q_y(C) = \frac{1}{Z_y} \mathbb{E} E_0 \left(1_C \, e^{-\sum_{j=0}^{\tau_y - 1} V(S_j, \omega)} 1_{\{\tau_y < \infty\}} \right).$$

When $C = A \times \Omega$ then the annealed measure of A, a subset of a path space, is just

$$\frac{1}{Z_y} \mathbb{E} \left(Q_y^\omega(A) Z_y^\omega \right).$$

Lyapunov exponents

The rates of decay of the quenched and annealed partition functions

$$\alpha_V(h) := -\lim_{r\to\infty}\frac{1}{r}\log Z^{\omega}_{[rh]} \quad \mathbb{P}\text{-a.s. and}$$
$$\beta_V(h) := -\lim_{r\to\infty}\frac{1}{r}\log Z_{[rh]}, \quad h\in\mathbb{R}^d,$$

known also as the quenched and annealed Lyapunov exponents, are well defined[1] non-random norms on $\mathbb{R}^d$. Functions $\lambda\mapsto\alpha_{\lambda+V}(h)$ and $\lambda\mapsto\beta_{\lambda+V}(h)$ are increasing and concave on $[0,\infty)$. Moreover, by Jensen's inequality, $\beta_V(h)\le\alpha_V(h)$ for all $h\in\mathbb{R}^d$.

Our results concern the question whether RWRP under the annealed path measure is ballistic, i.e. whether there is a constant $B\in(0,\infty)$ such that $E_{Q_y}(\tau_y)\le B\|y\|$.

[1] For the existence of $\alpha_V(\cdot)$ it is sufficient to assume that $\mathbb{E}V<\infty$.

Main results: $d = 1$

Theorem (EK, T. Mountford, 2011[2])

Let $d = 1$ *and* $\mathbb{P}(V(0) \in (0, \infty)) > 0$. *Then there is a constant* $v_a \in (0, 1)$ *such that*

$$\lim_{y \to \infty} \frac{E_{Q_y}(\tau_y)}{y} = \frac{1}{v_a}. \quad \textit{Moreover,} \quad \left. \frac{d\beta_{\lambda+V}(1)}{d\lambda} \right|_{\lambda=0+} = \frac{1}{v_a}.$$

The existence of the asymptotic quenched speed, v_q, is well-known for Brownian motion in a Poissonian potential[3].

Remark. The condition $\mathbb{P}(V(0) \in (0, \infty)) > 0$ is necessary for the result to hold. It is not difficult to show that if $\mathbb{P}(V(0) = 0) = 1 - \mathbb{P}(V(0) = \infty) \in (0, 1)$ then

$$\lim_{y \to \infty} y^{-2} E_{Q_y}(\tau_y) = 1/3.$$

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[2] arXiv:1103.0515v1 [math.PR]

[3] A.-S.Sznitman, Brownian Motion, Obstacles, and random Media, 1998.

Main results: $d \geq 2$

Theorem (EK, T. Mountford, 2011[4])

Let $d \geq 2$. Then there is a constant $B \in (0, \infty)$ such that

$$\limsup_{\|y\| \to \infty} \frac{E_{Q_y}(\tau_y)}{\|y\|} \leq B.$$

In particular, for every unit vector $s \in \mathbb{R}^d$

$$\frac{d\beta_{\lambda+V}(s)}{d\lambda}\bigg|_{\lambda=0+} \leq \limsup_{r \to \infty} \frac{E_{Q_{[rs]}}(\tau_{[rs]})}{[rs]} \leq B.$$

The corresponding quenched result in a continuous setting is known[5].

[4] arXiv:1103.0515v1 [math.PR]

[5] A.-S.Sznitman, Crossing velocities and random lattice animals, 1995.

After we submitted our paper we learned about an independent work by D. Ioffe and Y. Velenik[6], which treats the case $d \geq 2$ under the annealed measure and shows by different methods the existence of the asymptotic speed v_a for every direction, LLN, and CLT.

[6] arXiv:1104.4615v1 [math.PR]

Bernoulli potentials (work in progress)

An interesting question is to study the dependence of the asymptotic crossing velocity, v_a, on the "strength" of the potential. It has been shown[7] that for every unit vector $s \in \mathbb{R}^d$

$$\lim_{\gamma \to 0} \frac{\alpha_{\gamma V}(s)}{\sqrt{\gamma}} = \lim_{\gamma \to 0} \frac{\beta_{\gamma V}(s)}{\sqrt{\gamma}} = \sqrt{2d\mathbb{E}(V(0))}.$$

Let us consider $d = 1$ and Bernoulli potentials $V^{p,\gamma}$ such that $\mathbb{P}(V^{p,\gamma}(0) = \gamma) = p = 1 - \mathbb{P}(V^{p,\gamma}(0) = 0)$. Then the above result suggests that $v_q^{p,\gamma}$ and $v_a^{p,\gamma}$ behave like $c(p)\sqrt{\gamma}$ for γ small (p fixed).

Can one find out the behavior at the other extreme, i.e. when $\gamma \to \infty$? What can be said about the regime when γ is fixed and $p \to 0$ ("sparse" potentials)? For $d = 1$ there is a striking difference between the quenched and annealed behavior. The question is completely open for $d \geq 2$.

[7] EK, T. Mountford, M. Zerner, PTRF, 2011.

Coincidence of Lyapunov exponents

Recently, several works[8] addressed the question about the equality of quenched and annealed Lyapunov exponents for small perturbations of a constant potential. In particular, it was shown that when $d \geq 4$ then for every $\lambda > 0$ there is a $\gamma^* > 0$ such that for all $\gamma \in (0, \gamma^*)$

$$\beta_{\lambda+\gamma V}(\cdot) \equiv \alpha_{\lambda+\gamma V}(\cdot). \quad (1)$$

A stronger result was shown to hold[9]: for every $\lambda > 0$ there is a $\gamma^* > 0$ such that for all $\gamma \in (0, \gamma^*)$ the ratio Z_y^ω / Z_y converges $\mathbb{P}$-a.s.. It is certainly an interesting question whether (1) and its stronger version can be extended up to $\lambda = 0$ and whether γ^* is locally uniform in λ on $[0, \infty)$.

[8] M. Flury, Ann. Prob., 2008; N. Zygouras, PTRF, 2008; N. Zygouras, arXiv:1009.2693v2 [math.PR].

[9] D. Ioffe, Y. Velenik, arXiv:1002.4289v2 [math.PR]

Large deviations rate functions

The extension mentioned above would also help to clarify the relationship between the quenched and annealed large deviations rate functions for random walks in a random potential conditioned to survive up to ***time*** n. Let

$$Q_n^\omega(A) = \frac{1}{Z_n^\omega} E_0\left(1_A e^{-\sum_{j=0}^n V(S_j,\omega)}\right), \; Q_n(A) = \frac{1}{Z_n} \mathbb{E}\left(Q_n^\omega(A) Z_n^\omega\right).$$

Under these measures the linear speed of the walk is 0. It has been shown[10] that for $q \in \mathbb{R}^d$ and n large

$$Q_n^\omega\left(\frac{S_n}{n} \sim q\right) \asymp e^{-nI(q)(1+o(1))}, \; I(q) = \sup_{\lambda \geq 0}(\alpha_{\lambda+V}(q) - \lambda);$$

$$Q_n\left(\frac{S_n}{n} \sim q\right) \asymp e^{-nJ(q)(1+o(1))}, \; J(q) = \sup_{\lambda \geq 0}(\beta_{\lambda+V}(q) - \lambda).$$

[10] M. Donsker, S.R.S. Varadhan; A.-S. Sznitman; M. Zerner; M. Flury

Large deviations rate functions

$$Q_n^\omega\left(\frac{S_n}{n}\sim q\right)\asymp e^{-nI(q)(1+o(1))},\ I(q)=\sup_{\lambda\geq 0}(\alpha_{\lambda+V}(q)-\lambda);$$
$$Q_n\left(\frac{S_n}{n}\sim q\right)\asymp e^{-nJ(q)(1+o(1))},\ J(q)=\sup_{\lambda\geq 0}(\beta_{\lambda+V}(q)-\lambda).$$

By our results, $\left.\frac{d\beta_{\lambda+V}}{d\lambda}\right|_{\lambda=0+}<\infty$. This implies that when $\|q\|\ll 1$ the supremum in

$$J(q)=\|q\|\sup_{\lambda\geq 0}\left(\beta_{\lambda+V}\left(\frac{q}{\|q\|}\right)-\frac{\lambda}{\|q\|}\right)$$

is attained at $\lambda=0$. Thus $J(q)\equiv\beta_V(q)$ for $\|q\|\ll 1$. Similarly, $I(q)\equiv\alpha_V(q)$ for $\|q\|\ll 1$. The coincidence of Lyapunov exponents would imply the coincidence of the large deviations rate functions in a small neighborhood of the origin.

Polymer measures

Recall the annealed path measure for walks conditioned to survive up to time n,

$$Q_n(A) = \frac{1}{Z_n} \mathbb{E} \int_A e^{-\sum_{j=0}^{n-1} V(S_j,\omega)} \, dP^0.$$

Let $\ell_n^x = \sum_{j=0}^{n-1} 1_{\{S_j = x\}}$. Then $\sum_{j=0}^{n-1} V(S_j) = \sum_{x\in\mathbb{Z}^d} \ell_n^x V(x)$ and

$$\mathbb{E} e^{-\sum_{j=0}^{n-1} V(S_j,\omega)} = \prod_{x\in\mathbb{Z}^d} \mathbb{E} e^{-\ell_n^x V(x,\omega)}.$$

If we set

$$\phi(\ell) = -\log \mathbb{E} e^{-\ell V(0)}$$

then

$$Q_n(A) = \frac{1}{Z_n} \int_A e^{-\sum_{x\in\mathbb{Z}^d} \phi(\ell_n^x)} \, dP^0.$$

Phase transition for attractive polymers, $d \geq 2$.

Function ϕ is a non-negative non-decreasing concave function. This corresponds to an "attractive" case. By subtracting a constant from V it can always be normalized to be sublinear: $\phi(\ell)/\ell \to 0$ as $\ell \to \infty$.

$$Q_n(A) = \frac{1}{Z_n} \int_A e^{-\sum_{x \in \mathbb{Z}^d} \phi(\ell_n^x)} \, dP^0$$

can be generalized by replacing P^0 with P^h, where $h \in \mathbb{R}^d$ is a "drift". Then a change of measure gives

$$Q_n^h(A) = \frac{1}{Z_n^h} \int_A e^{-\sum_{x \in \mathbb{Z}^d} \phi(\ell_n^x) + h \cdot S_{n-1}} \, dP^0.$$

Phase transition for attractive polymers, $d \geq 2$.

$$Q_n^h(A) = \frac{1}{Z_n^h} \int_A e^{-\sum_{x \in \mathbb{Z}^d} \phi(\ell_n^x) + h \cdot S_{n-1}} \, dP^0.$$

It is known[10] that there is a set $K_a \subset \mathbb{R}^d$ containing the origin such that for every $h \in K_a$, S_n is sub-ballistic under Q_n^h and every $h \in \overline{K}_a^c$ it is ballistic. The results on crossing velocities for $d \geq 2$ imply[11] that this phase transition is of the first order: as $h \in \overline{K}_a^c$ approaches ∂K_a the speed approaches $v_a \neq 0$. Set K_a is convex and is exactly the set where $J(q) = \beta_V(q)$. The quenched situation was also addressed[12] but the existence of the asymptotic speed (I believe) has not been established.

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[10] A.-S. Sznitman, 1995; M. Flury, 2007

[11] D. Ioffe, Y. Velenik, 2011

[12] A.-S. Sznitman, Ann. Prob. 1995; M. Flury, ECP, 2008

Open (very hard) problem: transversal fluctuations

Let $d \geq 2$ and $y = Le_1$, $L \in \mathbb{Z}$. Consider the cylinder of width L^κ, $C(L,\kappa)$, around the first coordinate axis and define the event

$$A(L,\kappa) = \{\text{all paths } 0 \leftrightarrow y \text{ that do not exit } C(L,\kappa)\}.$$

Let

$$\zeta = \inf\{\kappa > 0 \,:\, \lim_{L\to\infty} \mathbb{E}(Q_y^\omega(A(L,\kappa)) = 1\}.$$

Similarly, one can consider walks conditioned to cross a hyperplane at distance L orthogonal to e_1.
It is known[13] that in the continuous setting (Brownian motion in a truncated and bounded away from 0 Poissonian potential conditioned to cross a hyperplane at distance L from the origin) the corresponding exponent $\zeta \geq 3/5$ for $d = 2$ and is at least $1/2$ for all $d \geq 3$. For the same model (for "point-to-point" crossings) $\zeta \leq 3/4$ in all dimensions. The conjectured value of ζ is $2/3$.

[13] M. Wüthrich, Ann. Prob., 1998; AIHP, 1998