Large deviations and fluctuation exponents for some polymer models

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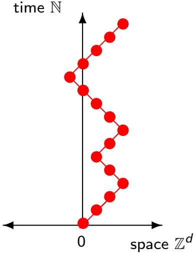
Directed polymer in a random environment



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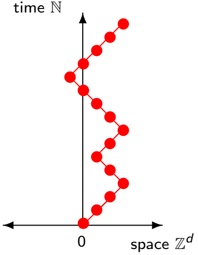

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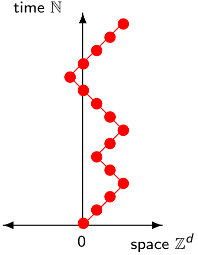

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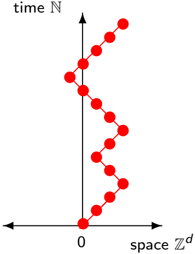

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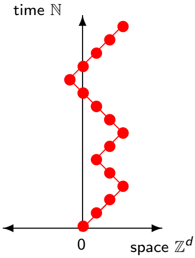

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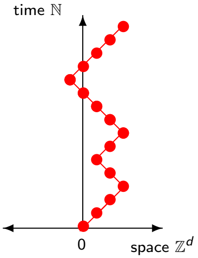

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$\mathbb{P}$ probability distribution on ω, often $\{\omega(x,t)\}$ i.i.d.

Key quantities again:

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- Dependence on β and d

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$g(\omega, z_{1,\ell})$ is a function on $\boldsymbol{\Omega}_\ell = \Omega \times \mathcal{R}^\ell$.

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Defines kernel p on $\mathbf{\Omega}_\ell$: $\;p(\eta, S_z\eta) = |\mathcal{R}|^{-1}$.

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$H_{\mathbb{P}}$ is convex but not lower semicontinuous.

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- IID directed + above moment $\Rightarrow$ $\Lambda(g)$ finite.

Quenched weak LDP (large deviation principle) under Q_n.

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IID environment, directed walk: full LDP holds.

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Early results: diffusive behavior for $d \geq 3$ and small $\beta > 0$:

1988 Imbrie and Spencer: $n^{-1}E^Q(|x(n)|^2) \to c \quad \mathbb{P}$-a.s.

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In the opposite direction: if $d = 1, 2$, or $d \geq 3$ and β large enough, then $\exists\ c > 0$ s.t.

$$\varlimsup_{n\to\infty} \max_z Q_n\{x(n) = z\} \geq c \quad \mathbb{P}\text{-a.s.}$$

(Carmona and Hu 2002, Comets, Shiga, and Yoshida 2003)

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Results: these exact exponents for three particular 1+1 dimensional models.

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- Gaussian RW in Gaussian potential: Petermann 2000 $\zeta \geq 3/5$, Mejane 2004 $\zeta \leq 3/4$
- Licea, Newman, Piza 1995-96: corresponding results for first passage percolation

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Next details on (3.i), then details on (1).

Hopf-Cole solution to KPZ equation

KPZ eqn for height function $h(t,x)$ of a 1+1 dim interface:

$$h_t \;=\; \tfrac{1}{2}\, h_{xx} \,-\, \tfrac{1}{2}\,(h_x)^2 \,+\, \dot W$$

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Bertini-Giacomin (1997): h can be obtained as a weak limit via a smoothing and renormalization of KPZ.

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$\zeta_\varepsilon(t, x)$ height process of weakly asymmetric simple exclusion s.t.

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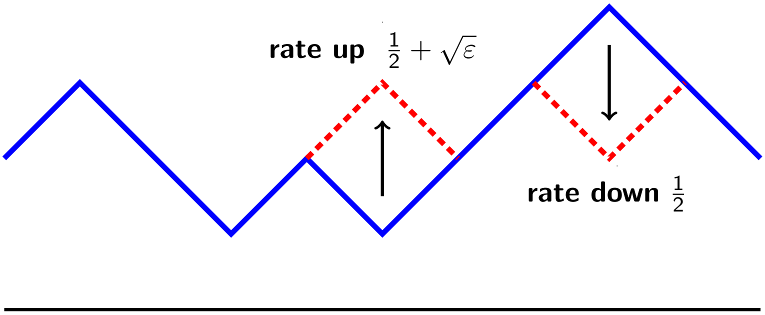

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Jumps:

$$\zeta_\varepsilon(x) \longrightarrow \begin{cases} \zeta_\varepsilon(x) + 2 & \text{with rate } \frac{1}{2} + \sqrt{\varepsilon} \quad \text{if } \zeta_\varepsilon(x) \text{ is a local min} \\ \zeta_\varepsilon(x) - 2 & \text{with rate } \frac{1}{2} \quad \text{if } \zeta_\varepsilon(x) \text{ is a local max} \end{cases}$$

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Theorem (Bertini-Giacomin 1997) As $\varepsilon \searrow 0$, $h_\varepsilon \Rightarrow h$

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(Second class particle estimate.)

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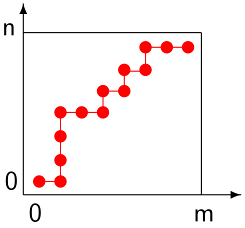
With some control over tails we arrive at the result:

$$\mathrm{Var}(h(t,0)) = \int |x|\,S(t,dx) \;\sim\; O(t^{2/3}).$$

1+1 dimensional lattice polymer with log-gamma weights

Fix both endpoints.



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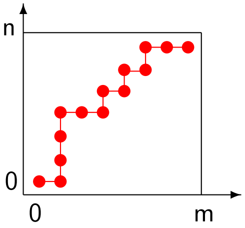

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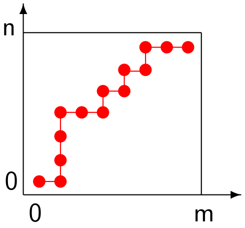

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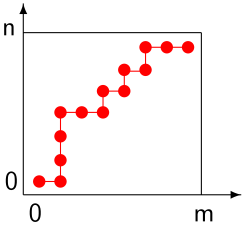

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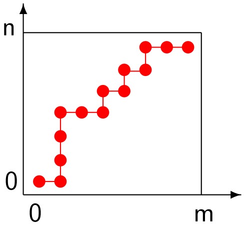

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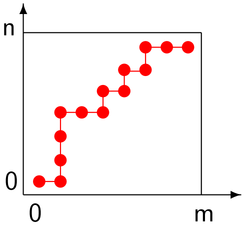

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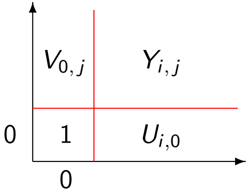

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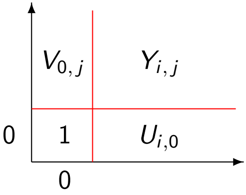

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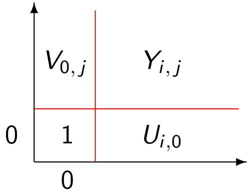

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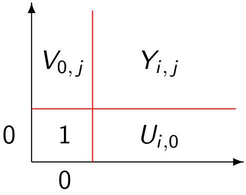

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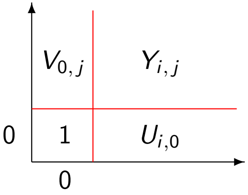

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Variance bounds for $\log Z$

With $0 < \theta < \mu$ fixed and $N \nearrow \infty$ assume

$$|\, m - N\Psi_1(\mu - \theta)\,| \le CN^{2/3} \quad \text{and} \quad |\, n - N\Psi_1(\theta)\,| \le CN^{2/3} \tag{1}$$

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Suppose $n = \Psi_1(\theta)N$ and $m = \Psi_1(\mu - \theta)N + \gamma N^{\alpha}$ with $\gamma > 0$, $\alpha > 2/3$. Then

$$N^{-\alpha/2}\Big\{\log Z_{m,n} - \mathbb{E}\big(\log Z_{m,n}\big)\Big\} \Rightarrow \mathcal{N}\big(0, \gamma\Psi_1(\theta)\big)$$

Fluctuation bounds for path

$v_0(j)$ = leftmost, $v_1(j)$ = rightmost point of $x_{\bullet}$ on horizontal line:

$$v_0(j) = \min\{i \in \{0, \dots, m\} : \exists k : x_k = (i, j)\}$$

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Assume (m, n) as previously and $0 < \tau < 1$. Then

$$\text{(a) } P\Big\{ v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \;\text{ or }\; v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3} \Big\} \le \frac{C}{b^3}$$

(b) $\forall \varepsilon > 0 \; \exists \delta > 0$ such that

$$\varlimsup_{N\to\infty} P\{ \; \exists k \text{ such that } |x_k - (\tau m, \tau n)| \le \delta N^{2/3} \; \} \le \varepsilon.$$

Results for log-gamma polymer summarized

With reciprocals of gammas for weights, both endpoints of the polymer fixed and the right boundary conditions on the axes, we have identified the one-dimensional exponents

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In both scenarios we have the upper bounds for both $\log Z$ and the path.

But currently do not have the lower bounds.

Results for log-gamma polymer summarized

With reciprocals of gammas for weights, both endpoints of the polymer fixed and the right boundary conditions on the axes, we have identified the one-dimensional exponents

$$\zeta = 2/3 \qquad \text{and} \qquad \chi = 1/3.$$

Next step is to

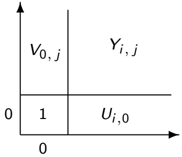
- eliminate the boundary conditions and
- consider polymers with fixed length and free endpoint

In both scenarios we have the upper bounds for both $\log Z$ and the path.

But currently do not have the lower bounds.

Next some key points of the proof.

Burke property for log-gamma polymer with boundary



Given initial weights $(i, j \in \mathbb{N})$:

$U_{i,0}^{-1} \sim \text{Gamma}(\theta)$, $\quad V_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta)$

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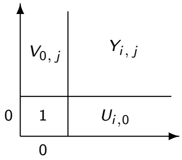

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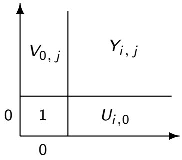

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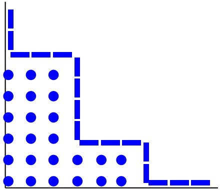
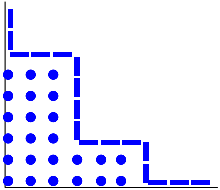
For an undirected edge f: $\quad T_f = \begin{cases} U_x & f = \{x - e_1, x\} \\ V_x & f = \{x - e_2, x\} \end{cases}$



down-right path (z_k) with edges $f_k = \{z_{k-1}, z_k\}$, $k \in \mathbb{Z}$

interior points $\mathcal{I}$ of path (z_k)



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Theorem

Variables $\{\, T_{f_k}, X_z : k \in \mathbb{Z},\, z \in \mathcal{I} \,\}$ are independent with marginals

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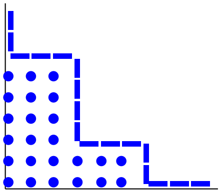

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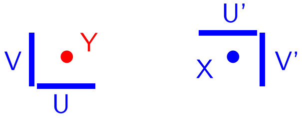
"Burke property" because the analogous property for last-passage is a generalization of Burke's Theorem for M/M/1 queues, via the last-passage representation of M/M/1 queues in series.

Proof of Burke property

Induction on $\mathcal{I}$ by flipping a growth corner:



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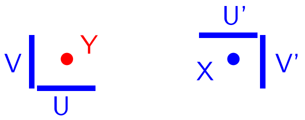

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Lemma. Given that (U, V, Y) are independent positive r.v.'s, $(U', V', X) \stackrel{d}{=} (U, V, Y)$ iff (U, V, Y) have the gamma distr's.

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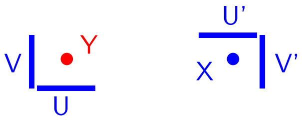

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This gives all (z_k) with finite $\mathcal{I}$. General case follows.

Proof of off-characteristic CLT

Recall that $\begin{cases} n = \Psi_1(\theta)N \\ m = \Psi_1(\mu-\theta)N + \gamma N^{\alpha} \end{cases}$ $\gamma > 0$, $\alpha > 2/3$.

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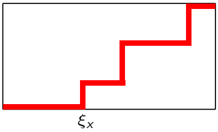
$$N^{-\alpha/2}\,\overline{\log Z_{m,n}} \;=\; N^{-\alpha/2}\,\overline{\log Z_{m_1,n}} \;+\; N^{-\alpha/2} \sum_{i=m_1+1}^{m} \overline{\log U_{i,n}}$$

First term on the right is $O(N^{1/3-\alpha/2}) \to 0$. Second term is a sum of order N^α i.i.d. terms. $\square$

Variance identity



Exit point of path from x-axis

$$\xi_x = \max\{k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k\}$$

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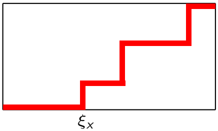

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$$L(\theta, x) = \int_0^x \bigl(\Psi_0(\theta) - \log y\bigr) x^{-\theta} y^{\theta-1} e^{x-y}\, dy$$

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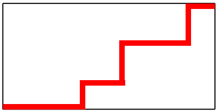

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Theorem. For the model with boundary,

$$\mathbb{V}\mathrm{ar}\bigl[\log Z_{m,n}\bigr] = n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\, E_{m,n}\Bigl[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Bigr]$$

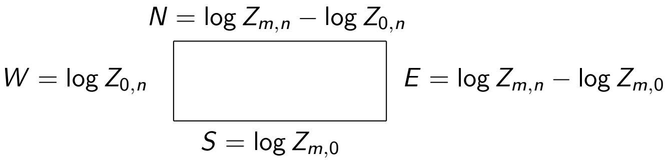
Variance identity, sketch of proof

$$N = \log Z_{m,n} - \log Z_{0,n}$$

$$W = \log Z_{0,n} \qquad E = \log Z_{m,n} - \log Z_{m,0}$$

$$S = \log Z_{m,0}$$

Variance identity, sketch of proof



$$
\begin{aligned}
\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] &= \mathbb{V}\mathrm{ar}(W+N) \\
&= \mathbb{V}\mathrm{ar}(W) + \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(W,N) \\
&= \mathbb{V}\mathrm{ar}(W) + \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(S+E-N,N) \\
&= \mathbb{V}\mathrm{ar}(W) - \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(S,N) \qquad (E, N \text{ ind.}) \\
&= n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\,\mathbb{C}\mathrm{ov}(S,N).
\end{aligned}
$$

To differentiate w.r.t. parameter θ of S while keeping W fixed, introduce a separate parameter ρ $(= \mu - \theta)$ for W.

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Differentiate: $\quad \displaystyle\frac{\partial}{\partial\theta}\log Z_{m,n}(\theta) \;=\; -\,E^{Q_{m,n}}\Big[\,\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Big].$

Together:

$$\begin{aligned}\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] &= n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\,\mathbb{C}\mathrm{ov}(S,N) \\ &= n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\,E_{m,n}\Bigg[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Bigg].\end{aligned}$$

This was the claimed formula. $\square$

Sketch of upper bound proof

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Since $H_\lambda(\eta) \le H_\theta(\eta)$,

$$Q^{\theta,\omega}\{\xi_x \ge u\} = \frac{1}{Z(\theta)} \sum_{x_\centerdot} \mathbf{1}\{\xi_x \ge u\} W(\theta) \;\le\; \frac{Z(\lambda)}{Z(\theta)} \cdot \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$

For $1 \le u \le \delta N$ and $0 < s < \delta$,

$$\mathbb{P}\big[Q^{\omega}\{\xi_x \ge u\} \ge e^{-su^2/N}\big] \le \mathbb{P}\bigg\{ \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \ge \alpha \bigg\} + \mathbb{P}\bigg(\frac{Z(\lambda)}{Z(\theta)} \ge \alpha^{-1} e^{-su^2/N} \bigg).$$

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Choose α right. Bound these probabilities with Chebychev which brings $\mathbb{V}\mathrm{ar}(\log Z)$ into play. In the characteristic rectangle $\mathbb{V}\mathrm{ar}(\log Z)$ can be bounded by $E(\xi_x)$. The end result is this inequality:

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Polymer in a Brownian environment

Environment: independent Brownian motions $B_1, B_2, \ldots, B_n$

Partition function (without boundary conditions):

$$Z_{n,t}(\beta) = \int\limits_{0<s_1<\cdots<s_{n-1}<t} \exp\Big[\beta\Big(B_1(s_1) \; + \; B_2(s_2) - B_2(s_1) \; +$$

$$+ \; B_3(s_3) - B_3(s_2) \; + \; \cdots + \; B_n(t) - B_n(s_{n-1}) \Big)\Big] \, ds_{1,n-1}$$