Properties of the Annealed Large Deviation Rate Function for a Random Walk in Random Environment

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RWRE in $\mathbb{Z}^d$

An *environment*: $\omega = \{\omega_x(\cdot)\}_{x\in\mathbb{Z}}$
$\omega_x(\cdot)$ is a measure on $\mathbb{Z}^d$ for each x.
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Annealed law $\mathbb{P}$: average over environments.

$$\mathbb{P}(G) := \int_\Omega P_\omega(G) dP(\omega)$$

Annealed Large Deviations

Theorem (Varadhan '03)

There exists a convex function $H(v)$ such that:
for all closed sets $F \subset \mathbb{R}^d$

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{X_n}{n} \in F\right) \leq -\inf_{v\in F} H(v),$$

and for all open sets $G \subset \mathbb{R}^d$

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{X_n}{n} \in G\right) \geq -\inf_{v\in G} H(v).$$

That is

$$\mathbb{P}(X_n \approx nv) = e^{-nH(v)+o(n)}.$$

Zero Set of the Rate Function

Possible drifts: $\mathcal{K} := conv\left(supp\{\sum_e \omega_0(e)e\}\right)$.
Nestling: $0 \in \mathcal{K}$.
Non-nestling: $0 \notin \mathcal{K}$.

Theorem (Varadhan '03)

The set $Z := \{v : H(v) = 0\}$ is either a single point or an interval containing the origin.
Moreover, $H(0) = 0$ if and only if the environment is nestling.

Varadhan's proof

X_n is not a Markov chain (long term memory).
Study the *comets* of the random walk:

$$W_n := (-X_n, -X_n + X_1, \ldots, -X_n + X_{n-1}, 0)$$

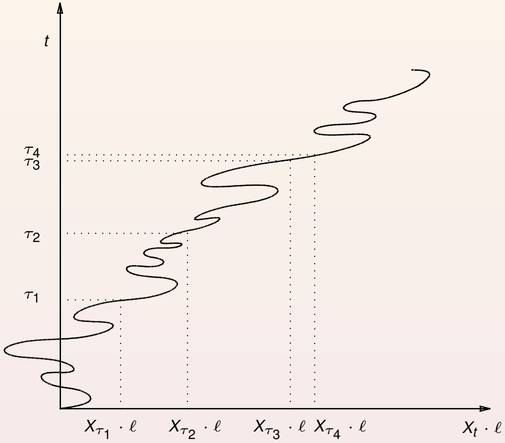
W_n is a Markov chain (on a horrible state space $\overline{W}$).
Obtain a LDP for the empirical distribution process

$$\mathcal{R}_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{W_n}$$

with rate function $\mathcal{J}(\mu)$.
Contract for LDP for $\frac{X_n}{n}$: $H(v) = \inf_{m(\mu)=v} \mathcal{J}(\mu)$.

Regeneration Times

Let $\ell \in \mathbb{R}^d$ with $\|\ell\|_2 = 1$.
Regeneration times (in direction ℓ):



Regeneration Times

$\left\{\left(X_{\tau_k} - X_{\tau_{k-1}}, \tau_k - \tau_{k-1}\right)\right\}_{k=1}^{\infty}$ is an independent sequence.

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Note that

$$v_P := \lim_{n\to\infty} \frac{X_n}{n} = \frac{\overline{\mathbb{E}} X_{\tau_1}}{\overline{\mathbb{E}} \tau_1}$$

The function $\mathcal{I}$

Define for $\lambda \in \mathbb{R}^{d+1}$

$$\Lambda(\lambda) := \log \overline{\mathbb{E}} e^{\lambda \cdot (X_{\tau_1}, \tau_1)},$$

and

$$\mathcal{I}(x,t) := \sup_{\lambda \in \mathbb{R}^{d+1}} \lambda \cdot (x,t) - \Lambda(\lambda).$$

Cramér's Theorem: $\left(\frac{X_{\tau_k}}{k}, \frac{\tau_k}{k}\right)$ satisfies a LDP under $\overline{\mathbb{P}}$ with rate function $\mathcal{I}$.

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- $\mathcal{I}(x,t)$ is convex.
- $\mathcal{I}(\overline{\mathbb{E}}\tau_1 v_P, \overline{\mathbb{E}}\tau_1) = 0$.
- $\Lambda(\lambda)$ is analytic in the interior of its domain and $\mathcal{I}(x,t)$ is convex and analytic in a neighborhood of $(\overline{\mathbb{E}}\tau_1 v_P, \overline{\mathbb{E}}\tau_1)$.

The function J

Define

$$J(v) := \inf_{s \in (0,1]} s\, \mathcal{I}\left(\frac{v}{s}, \frac{1}{s} \right).$$

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Theorem (P., Zeitouni '08)

Assume the law P is non-nestling. Then $H(v) = J(v)$ for all v in a neighborhood of v_P.

Idea of the proof

It is enough to prove

$$\lim_{\delta\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\| \frac{X_n}{n} - v \| < \delta \right) \leq -J(v),$$

and

$$\lim_{\delta\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\| \frac{X_n}{n} - v \| < \delta \right) \geq -J(v).$$

For convenience we'll work with $\overline{\mathbb{P}}$ instead of $\mathbb{P}$.

Lower bound

Fix $s \in (0, 1]$, and let $k = sn$.

$$\begin{aligned}
&\frac{1}{n} \log \overline{\mathbb{P}}(\|X_n - nv\| < 2\delta n) \\
&\quad \geq \frac{1}{n} \log \overline{\mathbb{P}}(\|X_{\tau_k} - nv\| < \delta n,\ |\tau_k - n| < \delta n) \\
&\quad = \frac{s}{k} \log \overline{\mathbb{P}} \left(\|\frac{X_{\tau_k}}{k} - \frac{v}{s}\| < \frac{\delta}{s},\ |\frac{\tau_k}{k} - \frac{1}{s}| < \frac{\delta}{s} \right)
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Limit as $k \to \infty$ and then $\delta \to 0$: $s\, \mathcal{I}\left(\frac{v}{s}, \frac{1}{s}\right)$.

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This lower bound is true for all $s \in (0,1]$ and so

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Note: Lower bound holds for any $v \cdot \ell > 0$.

Upper bound

First, note that by Chebychev's inequality

$$\overline{\mathbb{P}}(X_{\tau_k} = x,\ \tau_k = t) \leq e^{-\lambda\cdot(x,t)}\overline{\mathbb{E}}e^{\lambda\cdot(X_{\tau_k},\tau_k)}$$
$$= e^{-\lambda\cdot(x,t)+k\Lambda(\lambda)} = e^{-k\left(\lambda\cdot(\frac{x}{k},\frac{t}{k})-\Lambda(\lambda)\right)}.$$

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True for any $\lambda \in \mathbb{R}^{d+1}$ thus

$$\overline{\mathbb{P}}(X_{\tau_k} = x,\ \tau_k = t) \le e^{-k\mathcal{I}(\frac{x}{k},\frac{t}{k})} = e^{-t\frac{k}{t}\mathcal{I}(\frac{x}{t}\frac{t}{k},\frac{t}{k})} \le e^{-tJ(\frac{x}{t})}.$$

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Would like to say that

$$\overline{\mathbb{P}}(X_n \approx nv) \leq C\,\overline{\mathbb{P}}(\exists k : X_{\tau_k} \approx nv,\ \tau_k \approx n).$$

Upper bound

Since P is non-nestling, τ_1 has exponential tails:

$$\overline{\mathbb{P}}(\tau_1 \geq \varepsilon n) \leq Ce^{-C\varepsilon n}.$$

Fix ε small.
Since $J(v_P) = 0$, $J(v) < C\varepsilon$ in a neighborhood of v_P.

Thus we may assume $\tau_k - \tau_{k-1} < \varepsilon n$ for all $k \leq n$.
Need an upper bound for

$$\overline{\mathbb{P}}(\exists k : \tau_k \in (n - \varepsilon n, n], \ \|X_n - nv\| < n\delta, \ \tau_{k+1} > n).$$

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$$\leq \sum_{k \leq n} \sum_{s \in [0,\varepsilon)} \overline{\mathbb{P}}(\tau_k = (1-s)n,\ \|X_{\tau_k} - nv\| < n(\delta + s))\overline{\mathbb{P}}(\tau_1 > ns)$$

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Claim: Since $J(v)$ is quadratic near v_P, for v near v_P

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\inf_{s\in[0,\varepsilon)} \inf_{\|x-v\|<\delta+s} (1-s)J\left(\frac{x}{1-s}\right) + Cs = \inf_{\|x-v\|<\delta} J(x).
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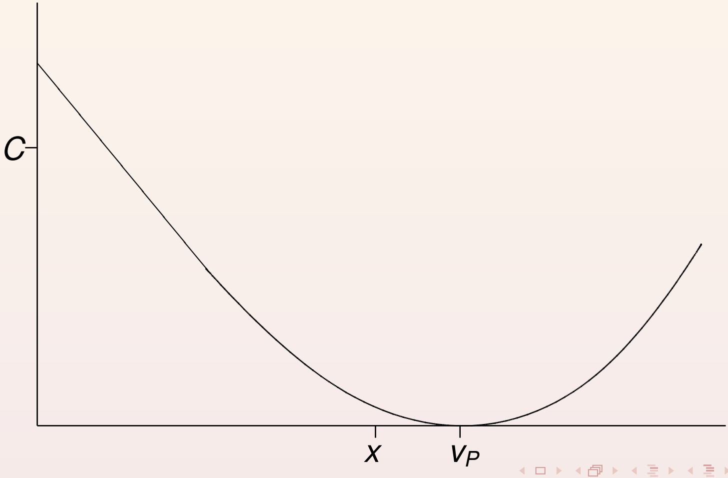

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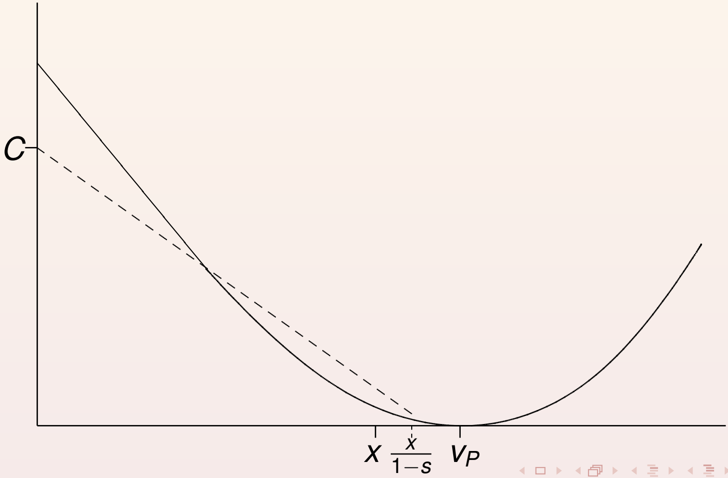

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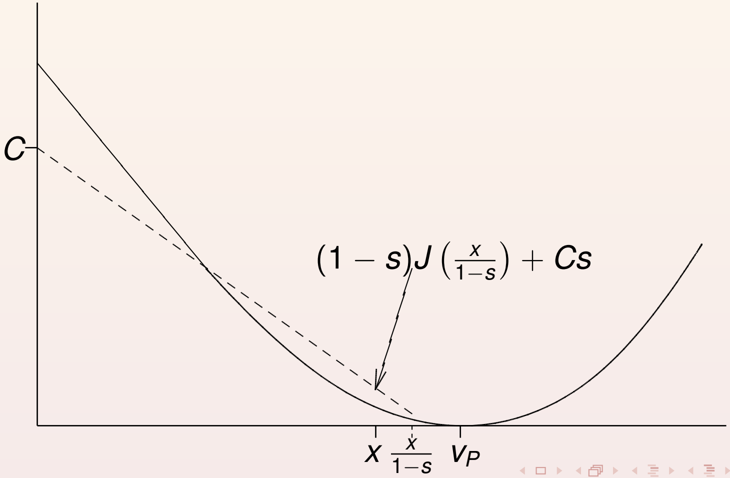

Large Deviations in One Dimension

Theorem (P., Zeitouni '08)

For any RWRE on $\mathbb{Z}$ in i.i.d. environment that is transient to $+\infty$ we have $H(v) = J(v)$ for all $v > 0$.