Bridges of Random Walks in a Random Environment

Jonathon Peterson

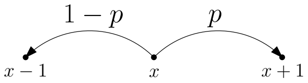
Cornell University
Department of Mathematics

Joint work with Nina Gantert

February 25, 2010

Bridges of Random Walks

S_n simple random walk.

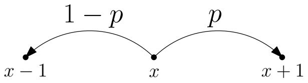


Question

What does the path of the random walk look like conditioned on $\{S_{2n} = 0\}$*?*

Bridges of Random Walks

S_n simple random walk.

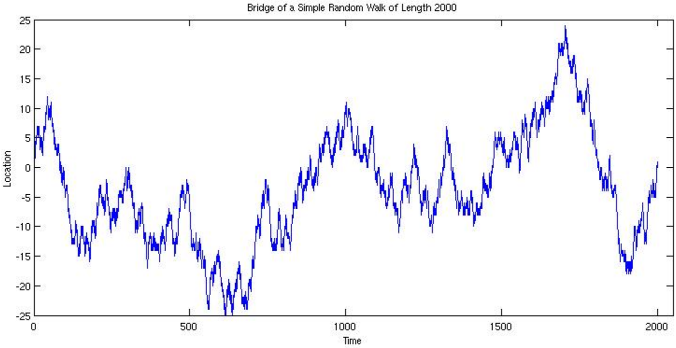


Question

What does the path of the random walk look like conditioned on $\{S_{2n} = 0\}$*?*

Distribution doesn't depend on p.

Scaled by $\sqrt{n}$, converges to Brownian Bridge.


Bridge of a Simple Random Walk of Length 2000
Location
Time
25
20
15
10
5
0
-5
-10
-15
-20
-25
0
500
1000
1500
2000

RWRE in $\mathbb{Z}$ with i.i.d. environment

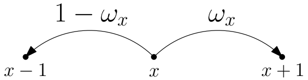
An *environment* $\omega = \{\omega_x\}_{x \in \mathbb{Z}} \in \Omega = [0,1]^{\mathbb{Z}}$.
P an i.i.d. product measure on Ω.

RWRE in $\mathbb{Z}$ with i.i.d. environment

An *environment* $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in \Omega = [0,1]^{\mathbb{Z}}$.
P an i.i.d. product measure on Ω.

Quenched law P_ω: fix an environment.
X_n a random walk: $X_0 = 0$, and

$$P_\omega(X_{n+1} = y+1 | X_n = y) := \omega_y$$

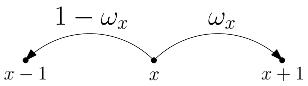


RWRE in $\mathbb{Z}$ with i.i.d. environment

An *environment* $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in \Omega = [0,1]^{\mathbb{Z}}$.
P an i.i.d. product measure on Ω.

Quenched law P_ω: fix an environment.
X_n a random walk: $X_0 = 0$, and

$$P_\omega(X_{n+1} = y+1 | X_n = y) := \omega_y$$



Averaged law $\mathbb{P}$: average over environments.

$$\mathbb{P}(G) := \int_\Omega P_\omega(G)\,dP(\omega)$$

Transience Criterion

A crucial statistic is:

$$\rho_x := \frac{1-\omega_x}{\omega_x}$$

Theorem (Solomon '75)

1. *If* $E_P(\log \rho_0) < 0$ *then,* $\lim_{n\to\infty} X_n = +\infty,\ \mathbb{P}-$ *a.s.*
2. *If* $E_P(\log \rho_0) > 0$ *then,* $\lim_{n\to\infty} X_n = -\infty,\ \mathbb{P}-$ *a.s.*
3. *If* $E_P(\log \rho_0) = 0$ *then,* X_n *is recurrent.*

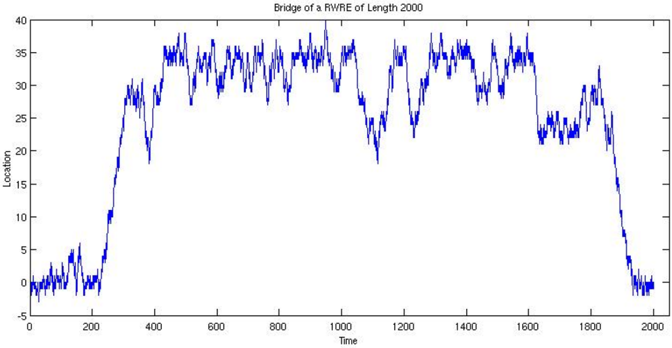
Bridges of RWRE

Assume i.i.d. environments.
Assume $E_P \log \rho < 0$ (transient to $+\infty$).

Question

What does the path of the random walk look like conditioned on $\{S_{2n} = 0\}$*?*

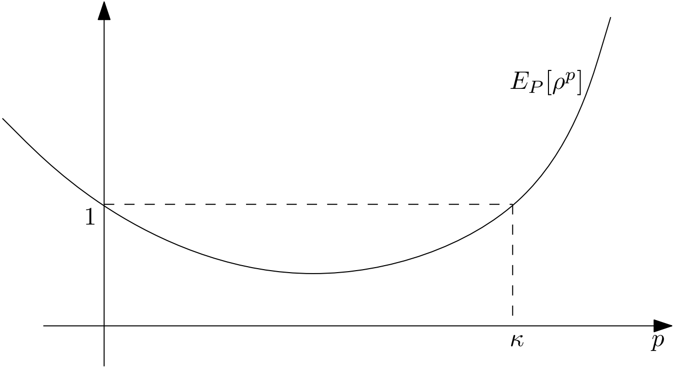
- *Does it depend on the environment?*
- *Does it look like a Brownian Bridge?*
- *Is the right scaling* $\sqrt{n}$*?*


Bridge of a RWRE of Length 2000
Location
Time
40
35
30
25
20
15
10
5
0
-5
0
200
400
600
800
1000
1200
1400
1600
1800
2000

Scale Parameter $\kappa(P)$ for Transient RWRE

Assume $E_P(\log \rho) < 0$ *(transience to the right)*.
Define $\kappa = \kappa(P)$ by

$$E_P \rho^{\kappa} = 1.$$



κ is related to the strength of the "traps".

Relevance of κ

Theorem (Law of Large Numbers - Solomon '75)

$\lim_{n\to\infty} \frac{X_n}{n} = v$, *and* $v > 0 \iff \kappa > 1$.

Theorem (Averaged Limit Laws - Kesten, Kozlov, Spitzer '75)

$$(a) \quad \kappa \in (0,1) \Rightarrow \quad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n}{n^\kappa} \le x \right) = 1 - L_{\kappa,b}(x^{-1/\kappa})$$

$$(b) \quad \kappa \in (1,2) \Rightarrow \quad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n v_P}{n^{1/\kappa}} \le x \right) = 1 - L_{\kappa,b}(-x)$$

$$(c) \quad \kappa > 2 \Rightarrow \quad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n v_P}{b\sqrt{n}} \le x \right) = \Phi(x)$$

where $L_{\kappa,b}$ is an κ-stable distribution function.

Case I: Positive and Negative Drifts

$\omega_{\text{min}} := \inf\{t : P(\omega_0 \le t) > 0\}$.

$E_P[\log \rho] < 0$ and $\omega_{\text{min}} < 1/2 \Rightarrow \kappa \in (0, \infty)$.

Case I: Positive and Negative Drifts

$\omega_{\min} := \inf\{t : P(\omega_0 \leq t) > 0\}$.

$E_P[\log \rho] < 0$ and $\omega_{\min} < 1/2 \Rightarrow \kappa \in (0, \infty)$.

Theorem (G. P. - '09)

Assume $E_P \log \rho < 0$ and $\omega_{\min} < 1/2$. Then,

$$\lim_{n\to\infty} P_\omega \left(\max_{k\leq 2n} |X_k| \leq n^\beta \,\middle|\, X_{2n} = 0 \right) = \begin{cases} 0 & \beta < \frac{\kappa}{\kappa+1} \\ 1 & \beta > \frac{\kappa}{\kappa+1}, \end{cases} \qquad P - a.s.$$

Suggests scaling of $n^{\kappa/(\kappa+1)}$.

- $\kappa < 1$ - Subdiffusive scaling
- $\kappa > 1$ - Superdiffusive scaling

Case II: Positive and Zero Drifts

Theorem (G. P. - '09)

Assume $\omega_{\min} = 1/2$ *and* $P(\omega_0 = 1/2) \in (0,1)$. *Then, for any* $\varepsilon > 0$

$$\lim_{n\to\infty} P_\omega \left(\max_{k \le 2n} |X_k| \le n^{1-\varepsilon} \;\middle|\; X_{2n} = 0 \right) = 0, \quad P - a.s.$$

and

$$\lim_{n\to\infty} P_\omega \left(\max_{k \le 2n} |X_k| \le \frac{n}{(\log n)^{2-\varepsilon}} \;\middle|\; X_{2n} = 0 \right) = 1, \quad P - a.s.$$

We suspect the proper scaling is $\frac{n}{(\log n)^2}$.

Case III: Strictly Positive Drifts

Theorem (G. P. - '09)

Assume $\omega_{\min} > 1/2$, $P(\omega_0 = \omega_{\min}) \in (0,1)$ *and* $P(\omega_0 \in (\omega_{\min}, \omega_{\min} + \delta)) = 1$ *for some* $\delta > 0$. *Then, for any* $\varepsilon > 0$

$$\lim_{n\to\infty} P_\omega \left(\max_{k\leq 2n} |X_k| \leq n^{1-\varepsilon} \,\middle|\, X_{2n} = 0 \right) = 0, \quad P - a.s.$$

and

$$\lim_{n\to\infty} P_\omega \left(\max_{k\leq 2n} |X_k| \leq \frac{n}{(\log n)^{2-\varepsilon}} \,\middle|\, X_{2n} = 0 \right) = 1, \quad P - a.s.$$

Again, we suspect the proper scaling is $\frac{n}{(\log n)^2}$.

Decay of $P_\omega(X_{2n} = 0)$

Case I: $\omega_{\min} < 1/2$

$$P_\omega(X_{2n} = 0) = \exp\left\{ -n^{\frac{\kappa}{\kappa+1} + o(1)} \right\}$$

Decay of $P_\omega(X_{2n} = 0)$

Case I: $\omega_{\min} < 1/2$

$$P_\omega(X_{2n} = 0) = \exp\left\{ -n^{\frac{\kappa}{\kappa+1} + o(1)} \right\}$$

Case II: $\omega_{\min} = 1/2$, and $\alpha = P(\omega_0 = 1/2) \in (0,1)$.

$$P_\omega(X_{2n} = 0) = \exp\left\{ -\frac{n}{(\log n)^2} \left(\frac{|\pi \log \alpha|^2}{4} + o(1) \right) \right\}$$

Decay of $P_\omega(X_{2n} = 0)$

Case I: $\omega_{\min} < 1/2$

$$P_\omega(X_{2n} = 0) = \exp\left\{ -n^{\frac{\kappa}{\kappa+1} + o(1)} \right\}$$

Case II: $\omega_{\min} = 1/2$, and $\alpha = P(\omega_0 = 1/2) \in (0,1)$.

$$P_\omega(X_{2n} = 0) = \exp\left\{ -\frac{n}{(\log n)^2} \left(\frac{|\pi \log \alpha|^2}{4} + o(1) \right) \right\}$$

Case III: $\omega_{\min} > 1/2$, and $\alpha = P(\omega_0 = \omega_{\min}) \in (0,1)$

$$P_\omega(X_{2n} = 0) = \exp\left\{ -I(0)n \qquad\qquad\qquad\qquad \right\}$$

where $I(0) = -\frac{1}{2}\log(4\omega_{\min}(1-\omega_{\min}))$.

Decay of $P_\omega(X_{2n} = 0)$

Case I: $\omega_{\min} < 1/2$

$$P_\omega(X_{2n} = 0) = \exp\left\{ -n^{\frac{\kappa}{\kappa+1} + o(1)} \right\}$$

Case II: $\omega_{\min} = 1/2$, and $\alpha = P(\omega_0 = 1/2) \in (0,1)$.

$$P_\omega(X_{2n} = 0) = \exp\left\{ -\frac{n}{(\log n)^2} \left(\frac{|\pi \log \alpha|^2}{4} + o(1) \right) \right\}$$

Case III: $\omega_{\min} > 1/2$, and $\alpha = P(\omega_0 = \omega_{\min}) \in (0,1)$

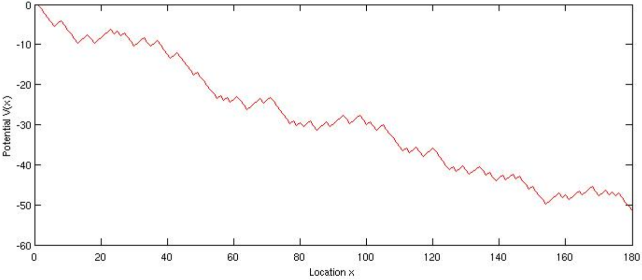
$$P_\omega(X_{2n} = 0) = \exp\left\{ -I(0)n - \frac{n}{(\log n)^2} \left(|\pi \log \alpha|^2 + o(1) \right) \right\}$$

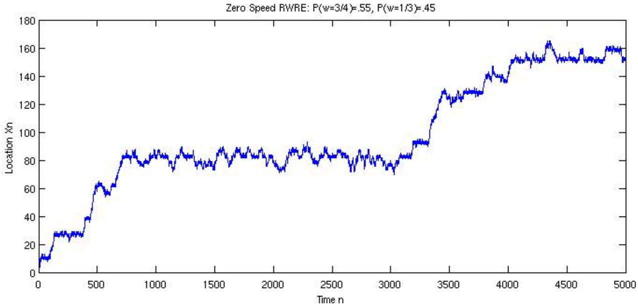
where $I(0) = -\frac{1}{2} \log(4\omega_{\min}(1 - \omega_{\min}))$.

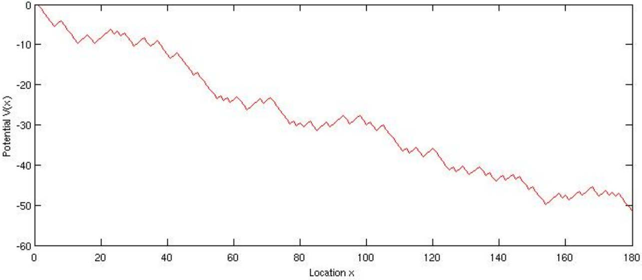
Potential and Traps

$$V(i) := \begin{cases} \sum_{k=0}^{i-1} \log \rho_k, & i > 0 \\ 0, & i = 0 \\ \sum_{k=i}^{-1} -\log \rho_k, & i < 0 \end{cases}$$

Trap: Atypical section where the potential is increasing.



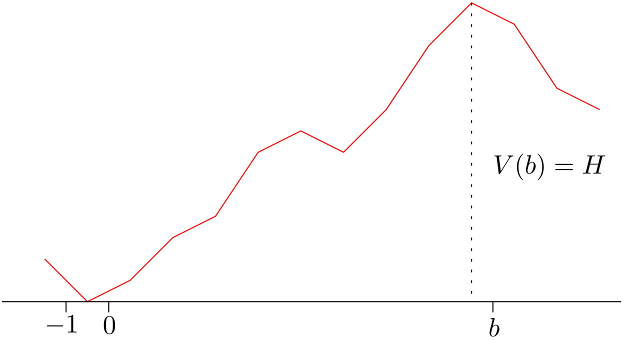

Zero Speed RWRE: P(w=3/4)=.55, P(w=1/3)=.45
180
160
140
120
100
80
60
40
20
0
Location Xn
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
Time n


0
-10
-20
-30
-40
-50
-60
Potential V(x)
0
20
40
60
80
100
120
140
160
180
Location x

Escaping Traps

Probability of escaping a trap of Height H.

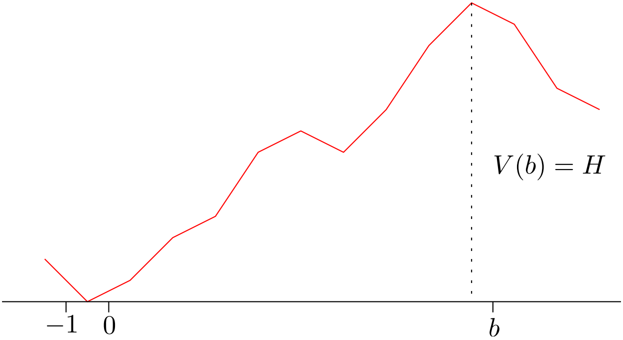
$$P_\omega(T_b < T_{-1}) = \frac{1}{1 + \sum_{j=1}^{b} e^{V(j)}}$$



Escaping Traps

Probability of escaping a trap of Height H.

$$P_\omega(T_b < T_{-1}) = \frac{1}{1 + \sum_{j=1}^{b} e^{V(j)}} \approx e^{-V(b)} = e^{-H}$$

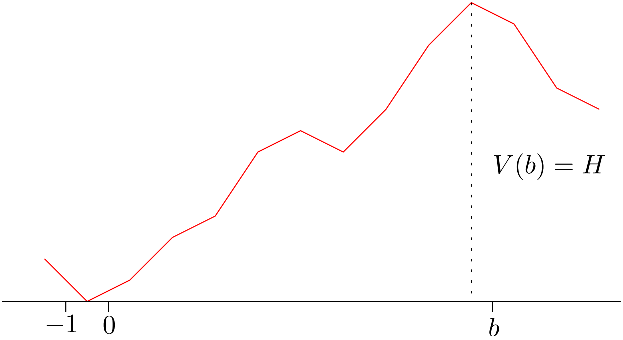


Escaping Traps

Probability of escaping a trap of Height H.

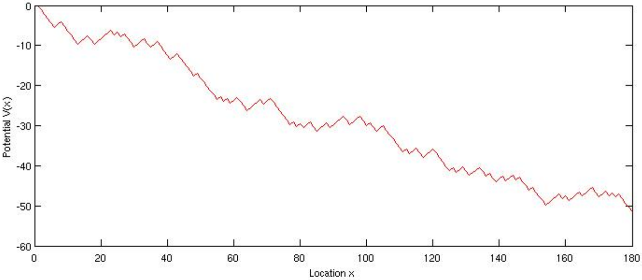
$$P_\omega(T_b < T_{-1}) = \frac{1}{1 + \sum_{j=1}^{b} e^{V(j)}} \approx e^{-V(b)} = e^{-H}$$

Time to escape trap of height $H \approx Exp(e^{-H})$.



Scaling of T_n

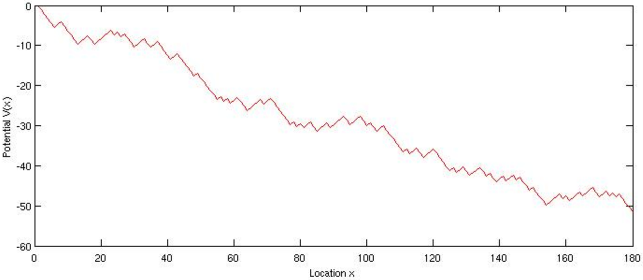
How long does the largest trap in $[0, n]$ contain the walk?



Scaling of T_n

How long does the largest trap in $[0, n]$ contain the walk?

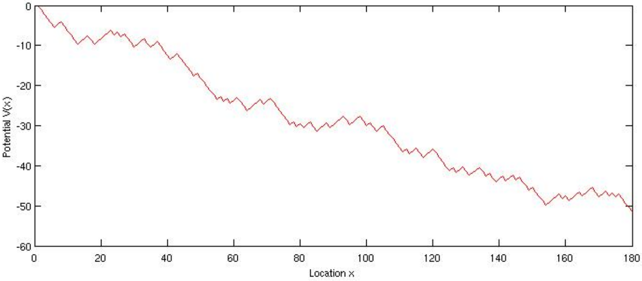
- Time to escape trap of height $H \approx Exp(e^{-H})$.



Scaling of T_n

How long does the largest trap in $[0, n]$ contain the walk?

- Time to escape trap of height $H \approx Exp(e^{-H})$.
- Largest uphill of $V(\cdot)$ in $[0, n]$ is $\sim \frac{1}{\kappa} \log n$ (Erdös & Renyi '70).

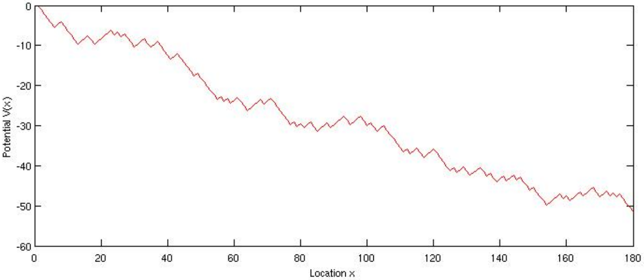


Scaling of T_n

How long does the largest trap in $[0, n]$ contain the walk?

- Time to escape trap of height $H \approx Exp(e^{-H})$.
- Largest uphill of $V(\cdot)$ in $[0, n]$ is $\sim \frac{1}{\kappa} \log n$ (Erdös & Renyi '70).

$\Rightarrow$ scaling of $n^{1/\kappa}$ in annealed limit laws of T_n.



Slowdowns - Large Deviations

Theorem (Gantert & Zeitouni '98)

Case I: *Positive and negative drifts. Let* $v \in (0, \mathrm{v}_P)$. *Then,* $P_\omega(X_n < nv) \approx e^{-n^{1-1/\kappa}}$*, in that for every* $\delta > 0$,

$$\limsup_{n\to\infty} \frac{\log P_\omega(X_n < nv)}{n^{1-1/\kappa+\delta}} = -\infty$$

$$\liminf_{n\to\infty} \frac{\log P_\omega(X_n < nv)}{n^{1-1/\kappa-\delta}} = 0$$

Slowdowns - Large Deviations

Theorem (Gantert & Zeitouni '98)

Case I: *Positive and negative drifts. Let* $v \in (0, \mathrm{v}_P)$. *Then,* $P_\omega(X_n < nv) \approx e^{-n^{1-1/\kappa}}$, *in that for every* $\delta > 0$,

$$\limsup_{n\to\infty} \frac{\log P_\omega(X_n < nv)}{n^{1-1/\kappa+\delta}} = -\infty$$

$$\liminf_{n\to\infty} \frac{\log P_\omega(X_n < nv)}{n^{1-1/\kappa-\delta}} = 0$$

Explanation:

- Deepest trap in $[0, nv]$ has depth $\approx \frac{1}{\kappa} \log n$
- Cost to stay in trap $\approx P(Exp(n^{-1/\kappa}) > n) = e^{-n^{1-1/\kappa}}$.

Slowdowns - Moderate Deviations

Theorem (Fribergh, Gantert, & Popov ('09))

Let $0 < \gamma < \kappa \wedge 1$*. Then,* $P_\omega(X_n < n^\gamma) = e^{-n^{\beta(\gamma)+o(1)}}$*, where*

$$\beta(\gamma) = \begin{cases} 1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1} \\ \frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1} \end{cases}$$

Slowdowns - Moderate Deviations

Theorem (Fribergh, Gantert, & Popov ('09))

Let $0 < \gamma < \kappa \wedge 1$*. Then,* $P_\omega(X_n < n^\gamma) = e^{-n^{\beta(\gamma)+o(1)}}$*, where*

$$\beta(\gamma) = \begin{cases} 1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1} \\ \frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1} \end{cases}$$

Strategy 1: Stay in deepest trap in $[0, n^\gamma]$.

Slowdowns - Moderate Deviations

Theorem (Fribergh, Gantert, & Popov ('09))

Let $0 < \gamma < \kappa \wedge 1$*. Then,* $P_\omega(X_n < n^\gamma) = e^{-n^{\beta(\gamma)+o(1)}}$*, where*

$$\beta(\gamma) = \begin{cases} 1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1} \\ \frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1} \end{cases}$$

Strategy 1: Stay in deepest trap in $[0, n^\gamma]$.

- Trap depth is $\approx \frac{1}{\kappa} \log n^\gamma = \frac{\gamma}{\kappa} \log n$
- Cost to stay in trap $\approx P(Exp(n^{-\gamma/\kappa}) > n) = e^{-n^{1-\gamma/\kappa}}$.

Slowdowns - Moderate Deviations

Theorem (Fribergh, Gantert, & Popov ('09))

Let $0 < \gamma < \kappa \wedge 1$. *Then,* $P_\omega(X_n < n^\gamma) = e^{-n^{\beta(\gamma)+o(1)}}$, *where*

$$\beta(\gamma) = \begin{cases} 1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1} \\ \frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1} \end{cases}$$

Strategy 1: Stay in deepest trap in $[0, n^\gamma]$.

- Trap depth is $\approx \frac{1}{\kappa} \log n^\gamma = \frac{\gamma}{\kappa} \log n$
- Cost to stay in trap $\approx P(Exp(n^{-\gamma/\kappa}) > n) = e^{-n^{1-\gamma/\kappa}}$.

Strategy 2: Look for a deeper trap farther out and then backtrack.

Slowdowns - Moderate Deviations

Theorem (Fribergh, Gantert, & Popov ('09))

Let $0 < \gamma < \kappa \wedge 1$. *Then,* $P_\omega(X_n < n^\gamma) = e^{-n^{\beta(\gamma)+o(1)}}$, *where*

$$\beta(\gamma) = \begin{cases} 1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1} \\ \frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1} \end{cases}$$

Strategy 1: Stay in deepest trap in $[0, n^\gamma]$.

- Trap depth is $\approx \frac{1}{\kappa} \log n^\gamma = \frac{\gamma}{\kappa} \log n$
- Cost to stay in trap $\approx P(Exp(n^{-\gamma/\kappa}) > n) = e^{-n^{1-\gamma/\kappa}}$.

Strategy 2: Look for a deeper trap farther out and then backtrack.

- Cost to trap in $[0, n^\beta]$ is $\approx e^{-n^{1-\beta/\kappa}}$.
- Cost to backtrack $\approx e^{-n^\beta}$.

Slowdowns - Moderate Deviations

Theorem (Fribergh, Gantert, & Popov ('09))

Let $0 < \gamma < \kappa \wedge 1$. *Then,* $P_\omega(X_n < n^\gamma) = e^{-n^{\beta(\gamma)+o(1)}}$, *where*

$$\beta(\gamma) = \begin{cases} 1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1} \\ \frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1} \end{cases}$$

Strategy 1: Stay in deepest trap in $[0, n^\gamma]$.

- Trap depth is $\approx \frac{1}{\kappa} \log n^\gamma = \frac{\gamma}{\kappa} \log n$
- Cost to stay in trap $\approx P(Exp(n^{-\gamma/\kappa}) > n) = e^{-n^{1-\gamma/\kappa}}$.

Strategy 2: Look for a deeper trap farther out and then backtrack.

- Cost to trap in $[0, n^\beta]$ is $\approx e^{-n^{1-\beta/\kappa}}$.
- Cost to backtrack $\approx e^{-n^\beta}$.
- Balance costs: $\beta = 1 - \beta/\kappa$ when $\kappa = \frac{\kappa}{\kappa+1}$.

Proof: Case I

$P(X_{2n} = 0) = e^{-n^{\kappa/(\kappa+1)+o(1)}}$.

Upper bound: $\beta > \frac{\kappa}{\kappa+1}$

$$P_\omega\left(\max_{k\leq 2n}|X_k| > n^\beta \mid X_{2n} = 0\right) \leq \frac{P_\omega(T_{-n^\beta} < \infty)}{P_\omega(X_{2n} = 0)} \leq \frac{e^{-n^\beta}}{e^{-n^{\kappa/(\kappa+1)}}}.$$

Lower bound: $\beta < \frac{\kappa}{\kappa+1}$

$$P_\omega\left(\max_{k\leq 2n}|X_k| \leq n^\beta \mid X_{2n} = 0\right) \leq \frac{P_\omega(\max_{k\leq 2n}|X_k| \leq n^\beta)}{P_\omega(X_{2n} = 0)} \leq \frac{e^{-n^{(1-\beta/\kappa)+o(1)}}}{e^{-n^{\kappa/(\kappa+1)}}}.$$

Slowdowns - Large Deviations

Theorem (Povel & Pisztora ('99))

Case II: *Positive and zero drift.* $\alpha = P(\omega_0 = \frac{1}{2})$. *Let* $v \in (0, \mathrm{v}_P)$. *Then,*

$$P_\omega(X_n < nv) = \exp\left\{ -\frac{n}{(\log n)^2} \left(\frac{|\pi \log \alpha|^2}{8}\left(1 - \frac{v}{\mathrm{v}_P}\right) + o(1) \right) \right\}$$

Slowdowns - Large Deviations

Theorem (Povel & Pisztora ('99))

Case II: *Positive and zero drift.* $\alpha = P(\omega_0 = \frac{1}{2})$. *Let* $v \in (0, \mathrm{v}_P)$. *Then,*

$$P_\omega(X_n < nv) = \exp\left\{ -\frac{n}{(\log n)^2} \left(\frac{|\pi \log \alpha|^2}{8}(1 - \frac{v}{\mathrm{v}_P}) + o(1) \right) \right\}$$

Explanation:

- Travel at typical speed for time $\frac{v}{\mathrm{v}_P} n$.
- Stay in long fair stretch for time $(1 - \frac{v}{\mathrm{v}_P})n$.

Slowdowns - Large Deviations

Theorem (Povel & Pisztora ('99))

Case II: *Positive and zero drift.* $\alpha = P(\omega_0 = \frac{1}{2})$. *Let* $v \in (0, \mathrm{v}_P)$. *Then,*

$$P_\omega(X_n < nv) = \exp\left\{ -\frac{n}{(\log n)^2} \left(\frac{|\pi \log \alpha|^2}{8}(1 - \frac{v}{\mathrm{v}_P}) + o(1) \right) \right\}$$

Explanation:

- Travel at typical speed for time $\frac{v}{\mathrm{v}_P} n$.
- Stay in long fair stretch for time $(1 - \frac{v}{\mathrm{v}_P})n$.
- Longest fair stretch $\sim \frac{1}{|\log \alpha|} \log n$.

Slowdowns - Large Deviations

Theorem (Povel & Pisztora ('99))

Case II: *Positive and zero drift.* $\alpha = P(\omega_0 = \frac{1}{2})$. *Let* $v \in (0, \mathrm{v}_P)$. *Then,*

$$P_\omega(X_n < nv) = \exp\left\{ -\frac{n}{(\log n)^2} \left(\frac{|\pi \log \alpha|^2}{8} (1 - \frac{v}{\mathrm{v}_P}) + o(1) \right) \right\}$$

Explanation:

- Travel at typical speed for time $\frac{v}{\mathrm{v}_P} n$.
- Stay in long fair stretch for time $(1 - \frac{v}{\mathrm{v}_P}) n$.
- Longest fair stretch $\sim \frac{1}{|\log \alpha|} \log n$.
- Small deviations for simple random walk: $r(N) = o(\sqrt{N})$

$$P\left(\max_{k \le N} |S_k| \le r(N) \right) \approx \exp\left\{ -\frac{\pi^2}{8} \frac{N}{r(N)^2} \right\}.$$

Proof: Case II

$$P(X_{2n} = 0) = \exp\left\{ -\frac{n}{(\log n)^2}\left(\frac{|\pi \log \alpha|^2}{4} + o(1) \right) \right\}.$$

Proof: Case II

$$P(X_{2n} = 0) = \exp\left\{ -\frac{n}{(\log n)^2}\left(\frac{|\pi \log \alpha|^2}{4} + o(1) \right) \right\}.$$

Upper bound: Backtracking $n/(\log n)^{2-\varepsilon}$ is too costly

$$P(T_{-n/(\log n)^{2-\varepsilon}} < \infty) \leq e^{-n/(\log n)^{2-\varepsilon}}$$

Proof: Case II

$$P(X_{2n}=0)=\exp\left\{-\frac{n}{(\log n)^2}\left(\frac{|\pi\log\alpha|^2}{4}+o(1)\right)\right\}.$$

Upper bound: Backtracking $n/(\log n)^{2-\varepsilon}$ is too costly

$$P(T_{-n/(\log n)^{2-\varepsilon}}<\infty)\le e^{-n/(\log n)^{2-\varepsilon}}$$

Lower bound: Confinement in $[-n^\gamma, n^\gamma]$

$$P\left(\max_{k\le 2n}|X_k|\le n^\gamma\right)=\exp\left\{-\frac{n}{(\log n)^2}\left(\frac{|\pi\log\alpha|^2}{4\gamma^2}+o(1)\right)\right\}$$

Longest fair stretch in $[-n^\gamma, n^\gamma]$ is $\sim \frac{\gamma}{|\log\alpha|}\log n$.

Case III - Positive Drifts

Change the environment

$$\tilde{\omega}_x = \frac{\rho_{\max}}{\rho_x + \rho_{\max}}.$$

$\tilde{\omega}$ has positive and zero drift.

Proposition (Gantert & P. ('09))

Let B_n = number of visits to sites with $\omega_x \neq \omega_{\min}$ in first $2n$ steps. Then, there exists a $c < 1$ such that for any $A \subset \{X_{2n} = 0\}$,

$$P_\omega(A) \approx e^{-2nI(0)} E_{\tilde{\omega}}[c^{B_n} \mathbf{1}_{\{A\}}]$$

Case III - Positive Drifts

Change the environment

$$\tilde{\omega}_x = \frac{\rho_{\max}}{\rho_x + \rho_{\max}}.$$

$\tilde{\omega}$ has positive and zero drift.

Proposition (Gantert & P. ('09))

Let B_n = number of visits to sites with $\omega_x \neq \omega_{\min}$ in first $2n$ steps. Then, there exists a $c < 1$ such that for any $A \subset \{X_{2n} = 0\}$,

$$P_\omega(A) \approx e^{-2nI(0)} E_{\tilde{\omega}}[c^{B_n} \mathbf{1}_{\{A\}}]$$

Ignoring B_n we obtain

$$P_\omega(X_{2n} = 0) \leq \exp\left\{ -2nI(0) - \frac{n}{(\log n)^2} \frac{|\pi \log \alpha|^2}{4} \right\}$$

Confinement Probabilities - Case III

Proposition (Gantert & P. ('09))

$$P_\omega(\max_{k\le 2n}|X_k| \le n^\gamma\,,\, X_{2n}=0)$$
$$= \exp\left\{ -I(0)n - \frac{n}{(\log n)^2}\left(\frac{|\pi \log \alpha|^2}{\gamma^2} + o(1) \right) \right\}$$

Confinement Probabilities - Case III

Proposition (Gantert & P. ('09))

$$P_\omega(\max_{k\le 2n}|X_k| \le n^\gamma\,,\, X_{2n}=0)$$
$$= \exp\left\{ -I(0)n - \frac{n}{(\log n)^2}\left(\frac{|\pi \log \alpha|^2}{\gamma^2} + o(1) \right) \right\}$$

Explanation:

- Convert to $\tilde{\omega}$.
- Longest fair stretch in $[-n^\gamma, n^\gamma]$ is $\sim \frac{\gamma}{|\log \alpha|} \log n$.

Confinement Probabilities - Case III

Proposition (Gantert & P. ('09))

$$P_\omega(\max_{k\leq 2n}|X_k| \leq n^\gamma \, , \, X_{2n} = 0)$$
$$= \exp\left\{ -I(0)n - \frac{n}{(\log n)^2}\left(\frac{|\pi \log \alpha|^2}{\gamma^2} + o(1) \right) \right\}$$

Explanation:

- Convert to $\tilde{\omega}$.
- Longest fair stretch in $[-n^\gamma, n^\gamma]$ is $\sim \frac{\gamma}{|\log \alpha|} \log n$.
- Restrict to $B_n \leq \frac{n}{(\log n)^{2-\varepsilon}}$: stay strictly inside fair stretch.