Current Fluctuations for a System of RWRE

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RWRE in $\mathbb{Z}$ with i.i.d. environment

An *environment* $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in \Omega = [0,1]^{\mathbb{Z}}$.
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Averaged law $\mathbb{P}$: average over environments.

$$\mathbb{P}(G) := \int_\Omega P_\omega(G) dP(\omega)$$

Recurrence / Transience

A crucial statistic is:

$$\rho_x := \frac{1-\omega_x}{\omega_x}$$

Theorem (Solomon '75)

1. *If* $E_P(\log \rho_0) < 0$ *then,* $\lim_{n\to\infty} X_n = +\infty, \ \mathbb{P} - a.s.$

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1. *If* $E_P(\log \rho_0) < 0$ *then,* $\lim_{n\to\infty} X_n = +\infty, \ \mathbb{P}-a.s.$
2. *Let* $E_P(\log \rho_0) < 0$*. Then,* $\mathbb{P}-a.s.$

   (a) $E_P\rho_0 < 1 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{X_n}{n} = \frac{1-E_P(\rho)}{1+E_P(\rho)} > 0$

   (b) $E_P\rho_0 \geq 1 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{X_n}{n} = 0.$

Denote $\lim_{n\to\infty} \frac{X_n}{n} =: \mathrm{v}_P$.

Central Limit Theorems

Theorem (Kesten, Kozlov, Spitzer '75, Iglehart '08, P. '08)

Let $E_P \rho_0^2 < 1$. Then, there exists constants $\sigma, \sigma_1 > 0$ such that

1. ***Averaged CLT:*** *For all $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n\mathrm{v}_P}{\sigma\sqrt{n}} \leq t \right) = \Phi(t).$$

2. ***Quenched CLT:*** *For all $t \in \mathbb{R}$, and $P-$a.e. environment ω,*

$$\lim_{n\to\infty} P_\omega\left(\frac{X_n - n\mathrm{v}_P + Z_n(\omega)}{\sigma_1\sqrt{n}} \leq t \right) = \Phi(t).$$

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Remark: If $E_P \log \rho_0 < 0$ and $E_P \rho_0^2 > 1$

- Averaged limiting distribution is non-Gaussian.
- No quenched limiting distribution possible.

Remarks on Quenched and Averaged CLT

Let $T_x := \inf\{n \geq 0 : X_n = x\}$. (Hitting time of x).

- Random centering term $Z_n(\omega) := \mathrm{v}_P(E_\omega T_{n\mathrm{v}_P} - \mathbb{E} T_{n\mathrm{v}_P})$, and

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Z_n(\omega)}{\sigma_2\sqrt{n}} \leq t \right) = \Phi(t)$$

- Scaling factors known explicitly:

Quenched CLT scaling $\sigma_1^2 = \mathrm{v}_P^3 E_P[\mathrm{Var}_\omega T_1]$

Z_n scaling $\sigma_2^2 = \mathrm{v}_P^2 \mathrm{Var}(E_\omega T_1)$

Averaged CLT scaling $\sigma^2 = \sigma_1^2 + \sigma_2^2$

- Quenched CLT: Convergence uniform over $t \in \mathbb{R}$.

System of Independent Walkers

$\eta_0(x) =$ Number of walkers starting at $x \in \mathbb{Z}$.

For $x \in \mathbb{Z}$ and $i \leq \eta_0(x)$, let $X_n^{x,i}$ be a RWRE starting at x.

All random walks evolve independently in the same environment.

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Assumptions on the initial conditions:
η_0 may depend on the environment ω.

- $P_\omega(\eta_0(x) = k) = P_{\theta^x \omega}(\eta_0(0) = k)$.
- Given ω, the $\eta_0(x)$ are independent (and independent of the RW)
- $E_P \left[E_\omega(\eta_0(0))^{2+\varepsilon} + \mathrm{Var}_\omega(\eta_0(0))^{2+\varepsilon} \right] < \infty$ for some $\varepsilon > 0$.

Uniform LLN

Recall: $\lim_{n\to\infty} \left| \frac{X_n}{n} - \mathrm{v}_P \right| = 0$, $\mathbb{P} - a.s.$

Theorem (P. '09)

Assume $\mathrm{v}_P > 0$ *and* $\mathbb{E}\eta_0 < \infty$. *Then, for any* $A < B$,

$$\lim_{n\to\infty} \max_{x\in(An,Bn]} \max_{i\leq\eta_0(x)} \left| \frac{X_n^{x,i} - x}{n} - \mathrm{v}_P \right| = 0, \quad \mathbb{P} - a.s.$$

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There exists an $s > 1$ such that

$$\mathbb{P}(|X_n - n\mathrm{v}_P| \geq \varepsilon n) \approx n^{1-s}.$$

$$P_\omega(|X_n - n\mathrm{v}_P| \geq \varepsilon n) \approx e^{-n^{1-1/s}}.$$

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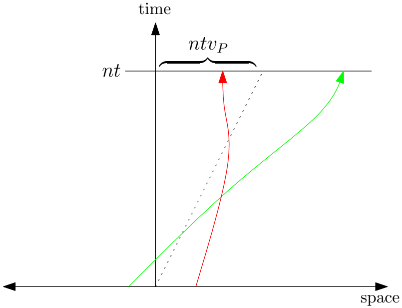
Need uniform quenched large deviations.

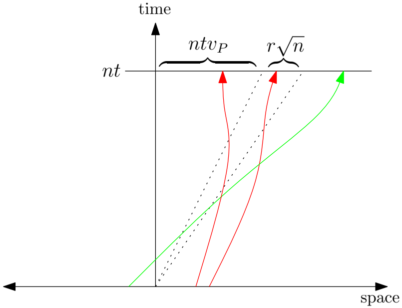
Current Process

$$Y_n(t,0) := \sum_{m>0} \sum_{i=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,i} \le n\mathrm{v}_P\} - \sum_{m\le 0} \sum_{i=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,i} > n\mathrm{v}_P\}$$



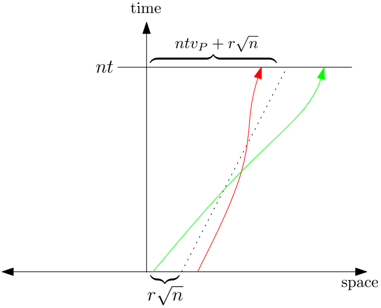
Space-Time Current Process

$$Y_n(t,r) := \sum_{m>0} \sum_{i=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,i} \le n\mathrm{v}_P + r\sqrt{n}\} - \sum_{m\le 0} \sum_{i=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,i} > n\mathrm{v}_P + r\sqrt{n}\}$$



Note that

$$Y_n(t,r) - Y_n(0,r) = \text{Current seen by observer starting at } r\sqrt{n}$$
$$\text{moving at speed } \mathrm{v}_P.$$



Current Process: Classical Random Walks

Theorem (Kumar '08)

For classical RW, the current process $n^{-1/4}(Y_n(t,r) - r\sqrt{n})$ converges in distribution to the mean-zero Gaussian process $V(t,r)$ with $E[V(s,q)V(t,r)] = \Gamma((s,q),(t,r))$, where

$$\begin{aligned}
&\Gamma((s,q),(t,r)) \\
&= \mu \int_{-\infty}^{\infty} \left(\mathbf{P}[B_{\sigma_1^2 s} \leq q - x]\mathbf{P}[B_{\sigma_1^2 t} > r - x] - \mathbf{P}[B_{\sigma_1^2 s} \leq q - x, B_{\sigma_1^2 t} > r - x] \right) dx \\
&\quad + \sigma_0^2 \left\{ \int_0^{\infty} \mathbf{P}[B_{\sigma_1^2 s} \leq q - x]\mathbf{P}[B_{\sigma_1^2 t} \leq r - x]\, dx + \int_{-\infty}^{0} \mathbf{P}[B_{\sigma_1^2 s} > q - x]\mathbf{P}[B_{\sigma_1^2 t} > r - x]\, dx \right\},
\end{aligned}$$

with $\mu = E[\eta_0(0)]$ and $\sigma_0^2 = Var(\eta_0(0))$.

RWRE Current Process: Quenched Mean

First order fluctuations of order $\sqrt{n}$ due to the environment.

Theorem (P., Seppäläinen '09)

For any $T < \infty$, $R < \infty$, and $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\sup_{|r|\le R,\ t\le T} \left| E_\omega Y_n(t,r) - \mu r\sqrt{n} - \mu Z_{nt}(\omega) \right| \ge \varepsilon\sqrt{n} \right) = 0.$$

Thus $\frac{E_\omega Y_n(t,r)}{\sqrt{n}}$ converges in distribution to $\mu\sigma_2 W(t) + \mu r$, where $W(\cdot)$ is a standard Brownian motion.

RWRE Current Process: $n^{1/4}$ fluctuations

$V_n(t,r) = Y_n(t,r) - E_\omega Y_n(t,r)$.

Theorem (P., Seppäläinen '09)

The finite dimensional distributions of the joint process

$$\left\{ \left(n^{-1/4} V_n(t,r), n^{-1/2} Z_{nt}(\omega) \right) \right\}_{t \geq 0,\ r \in \mathbb{R}}$$

converge to those of the process $(V(t,r), Z(t))$ *defined by*

- $Z(\cdot) = \sigma_2 W(\cdot)$, *where* $W(\cdot)$ *is a standard Brownian motion.*
- *Given* $Z(\cdot)$, $V(t,r)$ *is the mean-zero Gaussian process with covariance*

$$E[V(s,q)V(t,r)|Z(\cdot)] = \Gamma\left((s, q + Z(s)), (t, r + Z(t))\right).$$

Proof: Quenched Mean

Let $W_n(t) := E_\omega Y_n(t,0)$.
Special case: Fix $t > 0$, $r = 0$ and inititial condition $\eta_0(m) \equiv 1$.

$$W_n(t) = \sum_{m>0} P_\omega(X_{nt}^m \leq nt\mathrm{v}_P) - \sum_{m\leq 0} P_\omega(X_{nt}^m > nt\mathrm{v}_P)$$

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Sequence of approximations:

$$\widetilde{W}_n(t) := \sum_{m>0} \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\theta^m\omega) - m}{\sqrt{n}}\right) - \sum_{m\le 0} \Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\theta^m\omega) - m}{\sqrt{n}}\right)$$

$$\widehat{W}_n(t) := \sum_{m>0} \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\omega) - m}{\sqrt{n}} + r\right) - \sum_{m\le 0} \Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\omega) - m}{\sqrt{n}} - r\right)$$

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Note: We may restrict sums to $m \in [-a(n)\sqrt{n}, a(n)\sqrt{n}]$ for $a(n) \to \infty$.

Riemann sum approximation

$$\begin{aligned}\frac{\widehat{W}_n(t)}{\sqrt{n}} &= \frac{1}{\sqrt{n}}\left(\sum_{m>0}\Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\omega)-m}{\sqrt{n}}\right) - \sum_{m\leq 0}\Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\omega)-m}{\sqrt{n}}\right)\right)\\ &\approx \int_0^\infty \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\omega)}{\sqrt{n}}-x\right) - \Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\omega)}{\sqrt{n}}-x\right) dx\\ &= \frac{Z_{nt}(\omega)}{\sqrt{n}}.\end{aligned}$$

Riemann sum approximation

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$$\approx \int_0^\infty \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\omega)}{\sqrt{n}} - x\right) - \Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\omega)}{\sqrt{n}} - x\right) dx$$
$$= \frac{Z_{nt}(\omega)}{\sqrt{n}}.$$

Remains to show

$$\lim_{n\to\infty} P(|W_n(t) - \widetilde{W}_n(t)| \geq \varepsilon\sqrt{n}) = 0$$

$$\lim_{n\to\infty} P(|\widetilde{W}_n(t) - \widehat{W}_n(t)| \geq \varepsilon\sqrt{n}) = 0$$

$W_n(t) \approx \widetilde{W}_n(t)$

$$D(n,\omega) := \sup_{t \in \mathbb{R}} \left| P_\omega \left(\frac{X_n - n\mathrm{v}_P + Z_n(\omega)}{\sqrt{n}} \leq t \right) - \Phi_{\sigma_1^2}(t) \right|$$

Quenched CLT implies $D(n,\omega) \to 0$ for $P - a.e.$ environment ω.

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Quenched CLT implies $D(n,\omega) \to 0$ for $P - a.e.$ environment ω.

$$\begin{aligned} P_\omega(X_{nt}^m \leq nt\mathrm{v}_P) &= P_{\theta^m\omega}(X_{nt} \leq nt\mathrm{v}_P - m) \\ &= P_{\theta^m\omega}\left(\frac{X_{nt} - nt\mathrm{v}_P + Z_{nt}(\theta^m\omega)}{\sqrt{n}} \leq \frac{Z_{nt}(\theta^m\omega) - m}{\sqrt{n}} \right) \\ &\approx \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\theta^m\omega) - m}{\sqrt{n}} \right). \end{aligned}$$

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So,

$$\frac{1}{\sqrt{n}} |W_n(t) - \widetilde{W}_n(t)| \leq \frac{1}{\sqrt{n}} \sum_{m=-a(n)\sqrt{n}}^{a(n)\sqrt{n}} D(nt, \theta^m\omega).$$

$\widetilde{W}_n(t) \approx \widehat{W}_n(t)$

$$|\widetilde{W}_n(t) - \widehat{W}_n(t)| \leq \sum_{m=-a(n)\sqrt{n}}^{a(n)\sqrt{n}} \left| \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\theta^m \omega) - m}{\sqrt{n}} \right) - \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\omega) - m}{\sqrt{n}} \right) \right|$$

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Since $|\Phi_{\sigma_1^2 t}(x) - \Phi_{\sigma_1^2 t}(y)| \leq C|x-y|$,

$$\frac{1}{\sqrt{n}} |\widetilde{W}_n(t) - \widehat{W}_n(t)| \leq \frac{Ca(n)}{\sqrt{n}} \max_{|m| \leq a(n)\sqrt{n}} \left| Z_{nt}(\theta^m \omega) - Z_{nt}(\omega) \right| .$$

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$Z_{nt}(\omega)$ is a stationary (ergodic) sum:

$$Z_{nt}(\omega) = \mathrm{v}_P \left(E_\omega T_{nt\mathrm{v}_P} - \mathbb{E} T_{nt\mathrm{v}_P}\right) = \mathrm{v}_P \sum_{k=0}^{nt\mathrm{v}_P - 1} \left(E_{\theta^k\omega} T_1 - \mathbb{E} T_1\right)$$

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$$Z_{nt}(\theta^m \omega) = \mathrm{v}_P \sum_{k=m}^{m + nt\mathrm{v}_P - 1} \left(E_{\theta^k \omega} T_1 - \mathbb{E} T_1 \right)$$

Proof: $n^{1/4}$ fluctuations

$$V_n(t,r) = Y_n(t,r) - E_\omega Y_n(t,r)$$

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V_n(t,r) &= Y_n(t,r) - E_\omega Y_n(t,r) \\
&= \sum_{m>0} \left\{ \sum_{i=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,i} \leq nt\mathrm{v}_P + r\sqrt{n}\} - E_\omega \eta_0(m) P_\omega(X_{nt}^m \leq nt\mathrm{v}_P + r\sqrt{n}) \right\} \\
&\quad + \sum_{m\leq 0} \left\{ \sum_{i=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,i} > nt\mathrm{v}_P + r\sqrt{n}\} - E_\omega \eta_0(m) P_\omega(X_{nt}^m > nt\mathrm{v}_P + r\sqrt{n}) \right\} \\
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$$

- Given ω, the $u(m)$ are inedependent and zero-mean.
- Use Lindberg-Feller techniques (quenched).

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V_n(t,r) &= Y_n(t,r) - E_\omega Y_n(t,r) \\
&= \sum_{m>0} \left\{ \sum_{i=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,i} \le nt\mathrm{v}_P + r\sqrt{n}\} - E_\omega \eta_0(m) P_\omega(X_{nt}^m \le nt\mathrm{v}_P + r\sqrt{n}) \right\} \\
&\quad + \sum_{m\le 0} \left\{ \sum_{i=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,i} > nt\mathrm{v}_P + r\sqrt{n}\} - E_\omega \eta_0(m) P_\omega(X_{nt}^m > nt\mathrm{v}_P + r\sqrt{n}) \right\} \\
&= \sum_{m\in\mathbb{Z}} u(m).
\end{aligned}
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- Given ω, the $u(m)$ are inedependent and zero-mean.
- Use Lindberg-Feller techniques (quenched).
- May restrict sum to $|m| \le a(n)\sqrt{n}$.

Proof: $n^{1/4}$ fluctuations

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- May restrict sum to $|m| \le a(n)\sqrt{n}$.
- Need to calculate $\mathrm{Cov}_\omega(V_n(s,q), V_n(t,r))$.

$\mathrm{Var}_\omega(V_n(t,r))$

$\mathrm{Var}_\omega(V_n(t,r)) = \sum_{m\in\mathbb{Z}} \mathrm{Var}_\omega(u(m))$.
If $m > 0$ then,

$$Var_\omega(u(m)) = Var_\omega\left(\sum_{i=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,i} \leq nt\mathrm{v}_P + r\sqrt{n}\}\right)$$

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&\quad + Var_\omega(\eta_0(m)) P_\omega(X_{nt}^m \le nt\mathrm{v}_P + r\sqrt{n})^2
\end{aligned}
$$

Similar for $m \le 0$.

$\text{Var}_\omega(V_n(t,r))$

$$\begin{aligned}
&\text{Var}_\omega(n^{-1/4}V_n(t,r)) \\
&= \frac{1}{\sqrt{n}}\Bigg\{ \sum_{m\in\mathbb{Z}} (E_\omega \eta_0(m)) P_\omega(X_{nt}^m \le nt\mathrm{v}_P + r\sqrt{n}) P_\omega(X_{nt}^m > nt\mathrm{v}_P + r\sqrt{n}) \\
&\qquad + \sum_{m>0} Var_\omega(\eta_0(m)) P_\omega(X_{nt}^m \le nt\mathrm{v}_P + r\sqrt{n})^2 + \sum_{m\le 0} Var_\omega(\eta_0(m)) P_\omega(X_{nt}^m > nt\mathrm{v}_P + r\sqrt{n})^2 \Bigg\}
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&\approx \mu\int_{-\infty}^{\infty}\Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}}{\sqrt{n}} - x + r\right)\Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}}{\sqrt{n}} + x - r\right)dx \\
&\qquad + \sigma_0^2\left\{\int_0^\infty \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}}{\sqrt{n}} - x + r\right)^2 dx + \int_{-\infty}^0 \Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}}{\sqrt{n}} + x - r\right)^2 dx\right\}
\end{aligned}
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Similar to the proof of $\frac{1}{\sqrt{n}}E_\omega Y_n(t,r) \approx \mu Z_{nt}(\omega)$.

$\mathrm{Var}_\omega(V_n(t,r))$

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\begin{aligned}
&\mathrm{Var}_\omega(n^{-1/4}V_n(t,r)) \\
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&\qquad + \sigma_0^2 \left\{ \int_0^{\infty} \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}}{\sqrt{n}} - x + r\right)^2 dx + \int_{-\infty}^{0} \Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}}{\sqrt{n}} + x - r\right)^2 dx \right\} \\
&= \Gamma\left(\left(t, r + \frac{Z_{nt}}{\sqrt{n}}\right), \left(t, r + \frac{Z_{nt}}{\sqrt{n}}\right) \right)
\end{aligned}
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Special Case I: Stationary Initial Condition

Assume $\mu = \sigma_0^2$. (True in the stationary case).
$n^{-1/4} V_n(\cdot, 0) \xrightarrow{\mathcal{D}} V(\cdot, 0)$.

In the Classical RW case:

- $\mathbb{E}[V(s,0)V(t,0)] = \frac{\mu\sigma_1}{\sqrt{2\pi}} \left(\sqrt{s} + \sqrt{t} - \sqrt{|s-t|} \right)$
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- $V(\cdot, 0)$ is a Gaussian process *given* $Z(\cdot)$.
- $V(\cdot, 0)$ is not a Gaussian process???

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- the same as classical RW current.