Electrical Network Calculations in Random Walks in Random Environments

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Much of this talk is based on the book **Random Walks and Electric Networks** by Peter G. Doyle and J. Laurie Snell.

Free download available at

http://arxiv.org/abs/math/0001057

Some of the graphics in this talk are also from this book.

Outline

1. Markov Chains
2. Electrical Networks and Reversible Markov Chains
3. Probability Calculations $\longrightarrow$ Electrical Calculations
4. Simple Random Walks
5. Random Walks in Random Environments

Markov Chains

Markov chain: a random process X_n with short term memory.

Movement governed by **transition probabilities:**

$$P(X_{n+1} = y | X_n = x) = p_{x,y}$$

Example:

$$p_{0,0} = 1$$
$$p_{i,i+1} = p_{i,i-1} = 1/2 \quad \text{if } i = 1, 2, 3, \text{ or } 4$$
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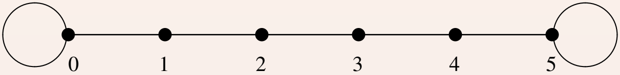

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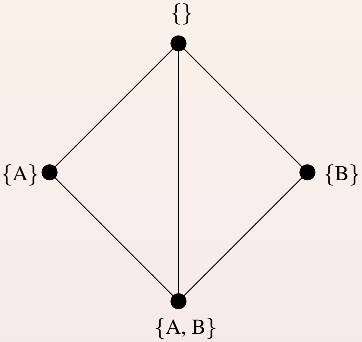
Examples

- **Population models:** $X_n =$ size of population on day n.
- **Stock market:** $X_n =$ price of stock on day n.
- **Spread of disease:** $X_n =$ subset of population infected on day n.



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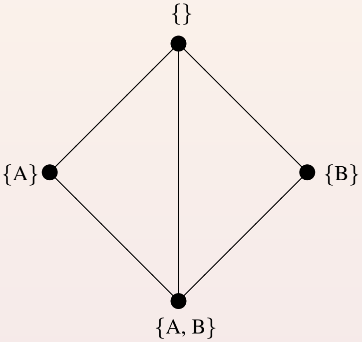

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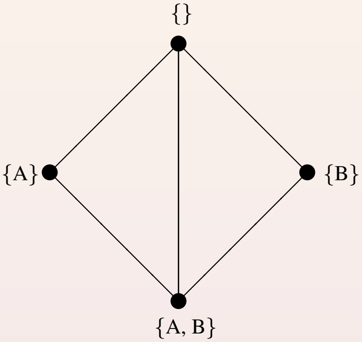
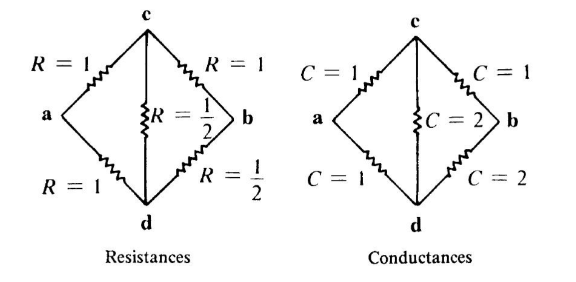


Electrical Networks

$R_{x,y}$ resistance of the edge from x to y.

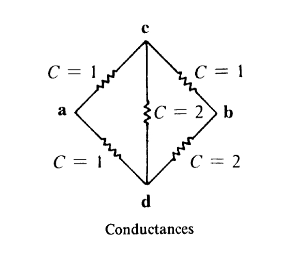
$C_{x,y} = \frac{1}{R_{x,y}}$ conductance of the edge from x to y.

$C_x = \sum_y C_{x,y}$. Total conductance at x.



Reversible Markov Chains

Given an electrical network, let $p_{x,y} = \frac{C_{x,y}}{C_x}$.



Example: $p_{b,c} = 1/3$ and $p_{b,d} = 2/3$.

These Markov chains are called *reversible Markov chains*.
When is a Markov chain reversible?

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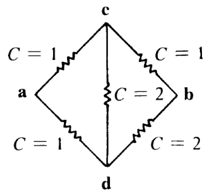

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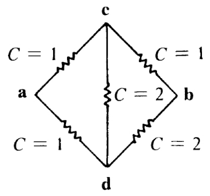

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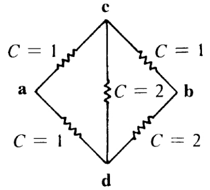

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Hitting Probabilities

Hitting times: $T_y := \inf\{n \geq 0 : X_n = y\}$.

We want to calculate **hitting probabilities:**

$$h(x) := P(T_a < T_b | X_0 = x) = P^x(T_a < T_b).$$

Obviously $h(a) = 1$ and $h(b) = 0$.
For $x \neq a, b$

$$h(x) = \sum_y p_{x,y} P^y(T_a < T_b) = \sum_y p_{x,y} h(y)$$

$h(x)$ is a **harmonic function.**

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Harmonic Functions

Graph G with edge weights $p_{x,y}$.
Subset of vertices B called the **boundary**.
$h(x)$ is (G, B, p)-harmonic if

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Theorem (Maximum (Minimum) Principle)

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An easy consequence of the Maximum Principle is:

Theorem (Uniqueness Principle)

If $h(x)$ and $v(x)$ are both (G, B, p)-harmonic functions with $h(x) = v(x)$ for all boundary points $x \in B$, then $v(x) = u(x)$ for all x.

Proof.

$u(x) = h(x) - v(x)$ is also (G, B, p)-harmonic.
$u(x) = 0$ for all $x \in B$.
By the Maximum (and minimum) principle, $u(x) = 0$ for all x. □

$h(x) = P(T_a < T_b | X_0 = x)$ is the unique $(G, \{a, b\}, p)$-harmonic function with boundary values $h(a) = 1$ and $h(b) = 0$.

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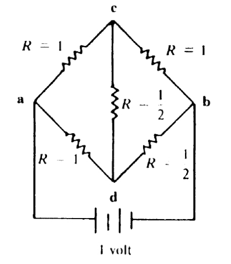
Voltage

Connect a 1V battery to nodes *a* and *b*.



$i_{x,y}$ is the current from x to y.
$v(x)$ is the voltage at node x. $v(a) = 1$ and $v(b) = 0$.

Ohm's Law: $i_{x,y} = (v(x) - v(y))C_{x,y}$.

Kirchoff's Current Law: $\sum_y i_{x,y} = 0$, for $x \neq a, b$.

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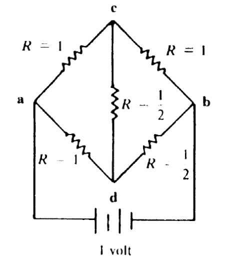

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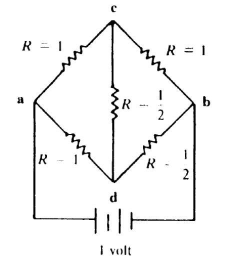

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For $x \notin \{a, b\}$,

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0 &= \sum_y i_{x,y} && \text{(Kirchoff's Law)} \\
&= \sum_y (v(x) - v(y)) C_{x,y} && \text{(Ohm's Law)} \\
&= v(x) C_x - \sum_y v(y) C_{x,y}.
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Therefore

$$v(x) = \sum_y \frac{C_{x,y}}{C_x} v(y), \quad \forall x \notin \{a, b\},$$

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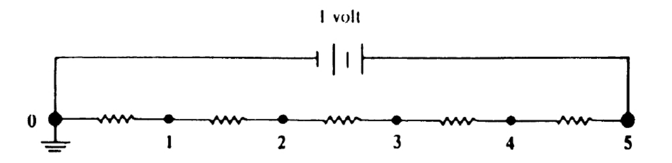
Hitting Probabilities and Voltage

For a Markov chain with transition probabilities $p_{x,y} = \frac{C_{x,y}}{C_x}$

$$h(x) = P^x(T_a < T_b) = v(x).$$

Example: Hitting Probabilities on an Interval



i is the total current flowing through the circuit.
$R(x \leftrightarrow y)$ is the **effective resistance** between x and y.
$C(x \leftrightarrow y) = \frac{1}{R(x \leftrightarrow y)}$ is the **effective conductance** between x and y.

By Ohm's Law: $v(x) = v(x) - v(b) = iR(x \leftrightarrow b)$.

Since $v(a) = 1$ this gives $i = \frac{1}{R(a \leftrightarrow b)}$, and therefore

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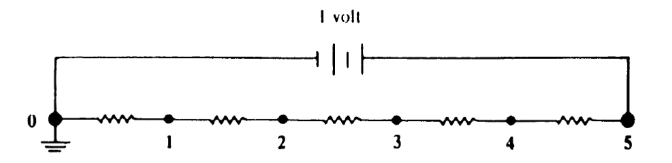

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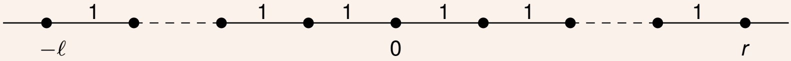
Simple Symmetric Random Walk

$$p_{i,i-1} = p_{i,i+1} = \frac{1}{2}$$

1 Ohm resistors at every edge.



$$P^0(T_r < T_{-\ell}) = \frac{R(0 \leftrightarrow -\ell)}{R(r \leftrightarrow -\ell)} = \frac{\ell}{r+\ell}.$$

$$P^0(T_r < \infty) = \lim_{\ell\to\infty} P^0(T_r < T_{-\ell}) = \lim_{\ell\to\infty} \frac{\ell}{r+\ell} = 1.$$

Simple symmetric random walk is **recurrent**.

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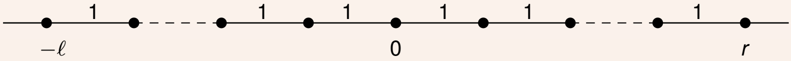

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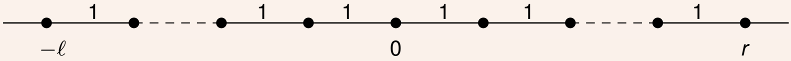

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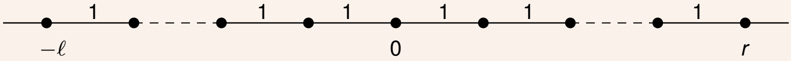

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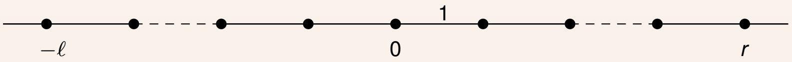
$$p_{i,i+1} = \omega \quad \text{and} \quad p_{i,i-1} = 1 - \omega$$



If $R_{0,1} = 1$, then $\omega = \frac{C_{1,2}}{1+C_{1,2}}$.

Thus $C_{1,2} = \frac{\omega}{1-\omega}$, or equivalently $R_{1,2} = \frac{1-\omega}{\omega}$.

Define $\rho := \frac{1-\omega}{\omega} < 1$.

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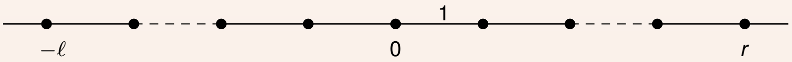

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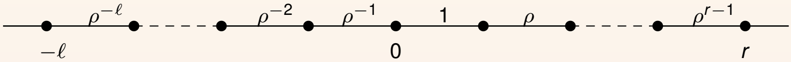

Then $R(0 \leftrightarrow r) = 1 + \rho + \cdots + \rho^{r-1} = \frac{1-\rho^r}{1-\rho}$.

and $R(0 \leftrightarrow -\ell) = \rho^{-1} + \rho^{-2} + \cdots + \rho^{-\ell} = \frac{\rho^{-\ell}-1}{1-\rho}$.

Therefore $R(r \leftrightarrow -\ell) = \frac{\rho^{-\ell}-\rho^r}{1-\rho}$.

$$P^0(T_r < \infty) = \lim_{\ell\to\infty} P^0(T_r < T_{-\ell}) = \lim_{\ell\to\infty} \frac{\rho^{-\ell}-1}{\rho^{-\ell}-\rho^r} = 1.$$

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Simple asymmetric random walk is **transient**.

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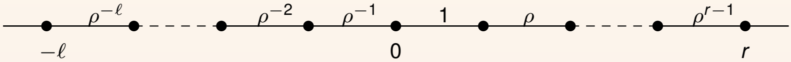

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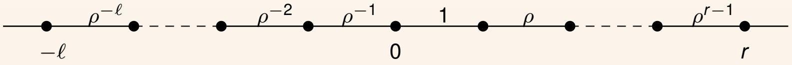

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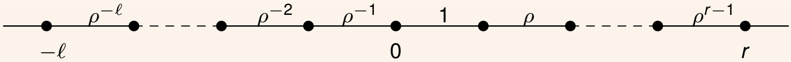

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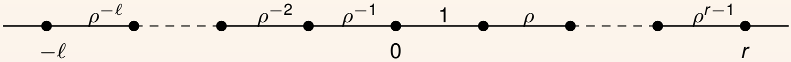

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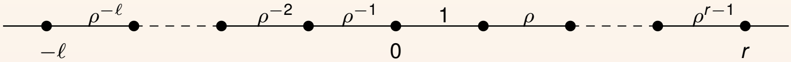

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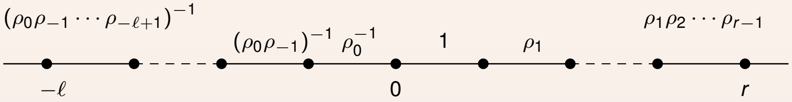

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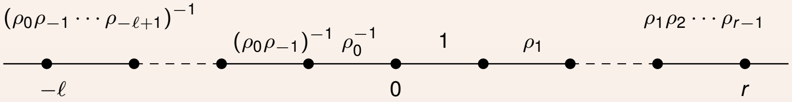

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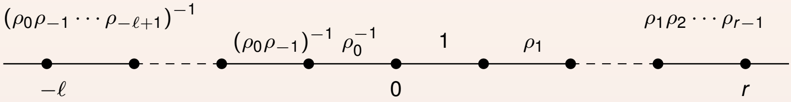

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&\approx e^{E(\log \rho_0)n} && \text{(Law of Large Numbers)}
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Example:

Suppose the distribution on environments is:

$$P\left(\omega_0 = \frac{3}{4}\right) = .39 \quad \text{and} \quad P\left(\omega_0 = \frac{1}{3}\right) = .61$$

Then

$$P(X_1 = 1) = .39\frac{3}{4} + .61\frac{1}{3} = \frac{119}{240} < \frac{1}{2},$$

and

$$E\rho_0 = .39\left(\frac{1-3/4}{3/4}\right) + .61\left(\frac{1-1/3}{1/3}\right) = \frac{27}{20} > 1.$$

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Transience with zero speed:
Can have $\lim_{n\to\infty} X_n = +\infty$ but $\lim_{n\to\infty} \frac{X_n}{n} = 0$.

Non-Gaussian Limiting Distributions:
For simple random walk, always have

$$\lim_{n\to\infty} P\left(\frac{X_n - nv}{\sigma\sqrt{n}} \le x\right) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

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