Quenched Limit Laws for Transient, One-Dimensional Random Walk in Random Environment

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RWRE in $\mathbb{Z}$ with i.i.d. environment

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P a product measure on Ω.

Quenched law P_ω: fix an environment.
X_n a random walk: $X_0 = x$, and

$$P_\omega(X_{n+1} = y+1 | X_n = y) := \omega_y$$

Annealed law $\mathbb{P}$: average over environments.

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Recurrence / Transience

A crucial statistic is:

$$\rho_x := \frac{1-\omega_x}{\omega_x}$$

Theorem (Solomon '75)

Transience or recurrence is determined by $E_P(\log \rho_0)$:

$$\begin{aligned}
(a)\quad & E_P(\log \rho_0) < 0 \Rightarrow \lim_{n\to\infty} X_n = +\infty, && \mathbb{P}-a.s. \\
(b)\quad & E_P(\log \rho_0) > 0 \Rightarrow \lim_{n\to\infty} X_n = -\infty, && \mathbb{P}-a.s. \\
(c)\quad & E_P(\log \rho_0) = 0 \Rightarrow X_n \text{ is recurrent}, && \mathbb{P}-a.s.
\end{aligned}$$

Law of Large Numbers

Assume $E_P(\log \rho) < 0$ *(transience to the right)*.
Assume $E_P \rho^s = 1$ for some $s > 0$.

Theorem (**LLN**, Solomon '75)

$\mathbb{P}$ – *a.s.:*

$$(a) \quad s > 1 \quad (E_P \rho < 1) \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{X_n}{n} = \frac{1 - E_P(\rho)}{1 + E_P(\rho)} > 0$$

$$(b) \quad s \leq 1 \quad (E_P \rho \geq 1) \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{X_n}{n} = 0$$

Denote $\lim_{n\to\infty} \frac{X_n}{n} =: v_P$.

Annealed Limit Laws

Theorem (Kesten, Kozlov, Spitzer '75)

There exists a constant b such that

$$(a) \quad s \in (0,1) \Rightarrow \quad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n}{n^s} \le x \right) = 1 - L_{s,b}(x^{-1/s})$$

$$(b) \quad s \in (1,2) \Rightarrow \quad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - nv_P}{n^{1/s}} \le x \right) = 1 - L_{s,b}(-x)$$

$$(c) \quad s > 2 \Rightarrow \quad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - nv_P}{b\sqrt{n}} \le x \right) = \Phi(x)$$

where $L_{s,b}$ is an s-stable distribution function.

Proof: First prove stable limit laws for hitting times

$$T_n := \inf\{k \ge 0 : X_k = n\}$$

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Characteristic Function of $L_{s,b}$:

$$\exp\left\{ -b|t|^s \left(1 - i \frac{t}{|t|} \tan(\pi s/2) \right) \right\}$$

Quenched Limit Laws (Gaussian Regime)

Theorem (Goldsheid '06, P. '06)

If $s > 2$ then

$$\lim_{n\to\infty} P_\omega\left(\frac{T_n - E_\omega T_n}{\sigma\sqrt{n}} \le x \right) = \Phi(x), \quad P - a.s.$$

where $\sigma^2 = E_P(Var_\omega T_1)$, and

$$\lim_{n\to\infty} P_\omega\left(\frac{X_n - nv_P + Z_n(\omega)}{v_P^{3/2}\sigma\sqrt{n}} \le x \right) = \Phi(x), \quad P - a.s.$$

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Question: What happens when $s < 2$?
Do we get quenched stable laws?

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Traps

Define the potential of the environmnet

$$V(i) := \begin{cases} \sum_{k=0}^{i-1} \log \rho_k, & i > 0 \\ 0, & i = 0 \\ \sum_{k=i}^{-1} - \log \rho_k, & i < 0 \end{cases}$$

Trap: An atypical section of environment where the potential is increasing.

Time to cross a trap is exponential in the height of the uphill.

Largest uphill of $V(\cdot)$ in $[0, n]$ is $\sim \frac{1}{s} \log n$ (Erdös & Renyi '70).

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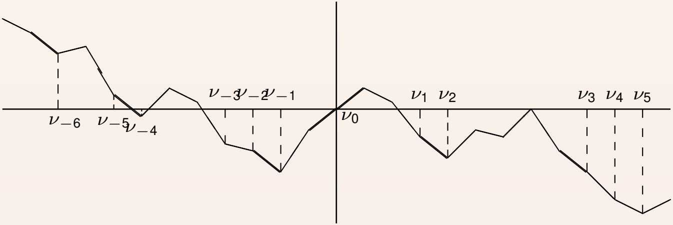
Blocks of the environment

Ladder locations $\{\nu_n\}$ defined by $\nu_0 = 0$,

$$\nu_n := \inf\{i > \nu_{n-1} : V(i) < V(\nu_{n-1})\}$$
$$\nu_{-n} := \sup\{j < \nu_{-n+1} : V(k) > V(j) \quad \forall k < j\}$$



Define a new measure on environments

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Under Q, the environment is stationary under shifts of the ν_i.

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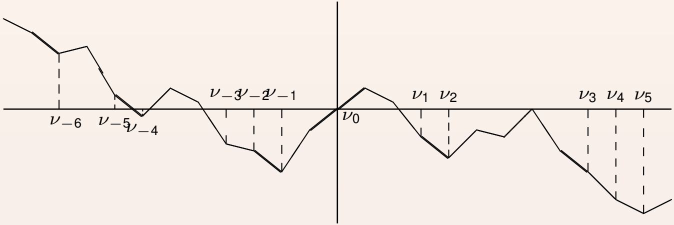

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Heuristics of Quenched Limit Laws

$$T_{\nu_n} = \sum_{i=1}^{n} (T_{\nu_i} - T_{\nu_{i-1}}) \stackrel{Law}{\approx} \sum_{i=1}^{n} \exp(\mu_{i,\omega})$$

where $\mu_{i,\omega} = E_\omega(T_{\nu_i} - T_{\nu_{i-1}}) \approx \sqrt{Var_\omega(T_{\nu_i} - T_{\nu_{i-1}})}$.

Quenched CLT? Only if

$$\lim_{n\to\infty} \max_{i\le n} \frac{\mu_{i,\omega}^2}{Var_\omega T_{\nu_n}} = 0, \quad P-a.s.$$

Exponential limit if

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Theorem (P. '07)

Assume $s < 2$. Then $\exists b > 0$ s.t.

$$\lim_{n\to\infty} Q\left(\frac{Var_\omega T_{\nu_n}}{n^{2/s}} \leq x \right) = L_{\frac{s}{2},b}(x).$$

α-stable process with $\alpha < 1$ has jumps.
This hints that when $s < 2$

$$\liminf_{n\to\infty} Q\left(\max_{i\leq n} \frac{\mu_{i,\omega}^2}{Var_\omega T_{\nu_n}} < \delta \right) > 0$$

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If $s < 2$ then P – a.s. there exist random subsequences $n_k = n_k(\omega)$, and $m_k = m_k(\omega)$ such that

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Contrast with the annealed results:

$$s \in (0,1) \Rightarrow \lim_{n\to\infty} \mathbb{P}\left(\frac{T_n}{n^{1/s}} \le x \right) = L_{s,b}(x)$$

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Theorem (P., Zeitouni '07)

If $s \in (0,1)$, then $P-$a.s. there exist random subsequences $n_k = n_k(\omega)$, $m_k = m_k(\omega)$, $t_k = t_k(\omega)$, and $u_k = u_k(\omega)$ s.t.

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not independent.

H_i the height of the block $[\nu_{i-1}, \nu_i)$. $Var_\omega T_\nu \approx (E_\omega T_\nu)^2 \approx e^{2H_1}$.
Iglehart ('72): $P(e^{H_i} > x) \sim Cx^{-s}$.

Binomial counting argument: Analyzing T_{ν_n}, only crossing times of blocks with $e^{H_i} > n^{(1-\varepsilon)/s}$ matter.

Blocks with $e^{H_i} > n^{(1-\varepsilon)/s}$ are well seperated
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Finding the subsequences:

With strictly positive probability (uniformly in n) find ...

One Large Block:

One block $i \leq n$ with

- $Var_\omega\left(T_{\nu_i} - T_{\nu_{i-1}}\right) \approx \mu_{i,\omega} > Mn^{2/s}$
- $\sum_{j \neq i} Var_\omega\left(T_{\nu_j} - Var_\omega T_{\nu_{j-1}}\right) \leq n^{2/s}$.

Then $\frac{\mu_{i,\omega}^2}{Var_\omega T_{\nu_n}} > \frac{M^2}{M^2+1} = 1 - \frac{1}{M^2+1}$.

No Dominating Blocks:

Subset $J \subset \{1, 2, \ldots, n\}$ of size $2k$ with

- $i \in J \Rightarrow Var_\omega\left(T_{\nu_i} - T_{\nu_{i-1}}\right) \approx \mu_{i,\omega}^2 \in [n^{2/s}, 2n^{2/s}]$
- $j \notin J \Rightarrow Var_\omega\left(T_{\nu_j} - T_{\nu_{j-1}}\right) < n^{2/s}$

Then $\frac{\mu_{i,\omega}^2}{Var_\omega T_{\nu_n}} < \frac{2n^{2/s}}{2kn^{2/s}} = \frac{1}{k}$.

Finding the subsequences:

With strictly positive probability (uniformly in n) find ...

One Large Block:

One block $i \leq n$ with

- $Var_\omega \left(T_{\nu_i} - T_{\nu_{i-1}} \right) \approx \mu_{i,\omega} > Mn^{2/s}$
- $\sum_{j \neq i} Var_\omega \left(T_{\nu_j} - Var_\omega T_{\nu_{j-1}} \right) \leq n^{2/s}$.

Then $\frac{\mu_{i,\omega}^2}{Var_\omega T_{\nu_n}} > \frac{M^2}{M^2+1} = 1 - \frac{1}{M^2+1}$.

No Dominating Blocks:

Subset $J \subset \{1, 2, \ldots, n\}$ of size $2k$ with

- $i \in J \Rightarrow Var_\omega \left(T_{\nu_i} - T_{\nu_{i-1}} \right) \approx \mu_{i,\omega}^2 \in [n^{2/s}, 2n^{2/s}]$
- $j \notin J \Rightarrow Var_\omega \left(T_{\nu_j} - T_{\nu_{j-1}} \right) < n^{2/s}$

Then $\frac{\mu_{i,\omega}^2}{Var_\omega T_{\nu_n}} < \frac{2n^{2/s}}{2kn^{2/s}} = \frac{1}{k}$.

References

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