Limiting Distributions and Large Deviations for Random Walks in Random Environments

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July 24, 2008

RWRE in $\mathbb{Z}^d$ with i.i.d. environment

An *environment* $\omega = \{\omega(x,y)\}_{x,y\in\mathbb{Z}^d}$, such that

$$\sum_{y\in\mathbb{Z}^d} \omega(x,y) = 1, \quad \forall x \in \mathbb{Z}^d.$$

$\{\omega(x,\cdot)\}_{x\in\mathbb{Z}^d}$ i.i.d. with distribution P.

Quenched law P_ω: fix an environment.
X_n a random walk: $X_0 = 0$, and

$$P_\omega(X_{n+1} = x + y | X_n = x) := \omega(x,y).$$

Annealed law $\mathbb{P}$: average over environments.

$$\mathbb{P}(G) := \int_\Omega P_\omega(G) dP(\omega)$$

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Definitions

Nearest neighbor:

$$\omega(x, y) > 0 \iff |y| = 1.$$

Elliptic:

$$P\left(\omega(x, y) \in (0, 1),\ \forall x \in \mathbb{Z}^d,\ \forall |y| = 1\right) = 1.$$

Uniformly Elliptic: $\exists \kappa > 0$ such that

$$P\left(\omega(x, y) \in [\kappa, 1 - \kappa],\ \forall x \in \mathbb{Z}^d,\ \forall |y| = 1\right) = 1.$$

Part I: Limit Distributions for Transient, One-Dimensional RWRE

RWRE in $\mathbb{Z}$: Recurrence / Transience

A crucial statistic is:

$$\rho_x := \frac{\omega(x,-1)}{\omega(x,1)}$$

Theorem (Solomon '75)

Transience or recurrence is determined by $E_P(\log \rho_0)$:

$$\begin{array}{lll} (a) & E_P(\log\rho_0) < 0 \Rightarrow \lim_{n\to\infty} X_n = +\infty, & \mathbb{P}-a.s. \\ (b) & E_P(\log\rho_0) > 0 \Rightarrow \lim_{n\to\infty} X_n = -\infty, & \mathbb{P}-a.s. \\ (c) & E_P(\log\rho_0) = 0 \Rightarrow X_n \text{ is recurrent}, & \mathbb{P}-a.s. \end{array}$$

RWRW in $\mathbb{Z}$: Law of Large Numbers

Assume $E_P(\log \rho) < 0$ *(transience to the right)*.
Assume $E_P \rho^s = 1$ for some $s > 0$.

Theorem (LLN, Solomon '75)

$\mathbb{P}-$*a.s.:*

$$(a) \quad s > 1 \quad (E_P \rho < 1) \quad \Rightarrow \quad \lim_{n\to\infty} \frac{X_n}{n} = \frac{1 - E_P(\rho)}{1 + E_P(\rho)} > 0$$

$$(b) \quad s \leq 1 \quad (E_P \rho \geq 1) \quad \Rightarrow \quad \lim_{n\to\infty} \frac{X_n}{n} = 0$$

Denote $\lim_{n\to\infty} \frac{X_n}{n} =: v_P$.

RWRE in $\mathbb{Z}$: Annealed Limit Laws

Theorem (Kesten, Kozlov, Spitzer '75)

There exists a constant b such that

$$(a) \quad s \in (0,1) \Rightarrow \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n}{n^s} \leq x \right) = 1 - L_{s,b}(x^{-1/s})$$

$$(b) \quad s \in (1,2) \Rightarrow \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n v_P}{n^{1/s}} \leq x \right) = 1 - L_{s,b}(-x)$$

$$(c) \quad s > 2 \Rightarrow \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n v_P}{b\sqrt{n}} \leq x \right) = \Phi(x)$$

where $L_{s,b}$ is an s-stable distribution function.

Characteristic Function of $L_{s,b}$:

$$\exp\left\{ -b|t|^s \left(1 - i\frac{t}{|t|}\tan(\pi s/2) \right) \right\}$$

RWRE in $\mathbb{Z}$: Annealed Limit Laws

Proof: First prove stable limit laws for hitting times

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Quenched Limit Laws (Gaussian Regime)

Theorem (Goldsheid '06, P. '06)

If $s > 2$ then

$$\lim_{n\to\infty} P_\omega\left(\frac{T_n - E_\omega T_n}{\sigma\sqrt{n}} \leq x \right) = \Phi(x), \quad P-a.s.$$

where $\sigma^2 = E_P(Var_\omega T_1)$, and

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$\{T_i - T_{i-1}\}_{i=1}^{\infty}$ are independent under P_ω. Lindberg-Feller $\Rightarrow$

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Define $X_t^* := \max\{X_n : n \leq t\}$. Then,

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Traps

Define the potential of the environmnet

$$V(i) := \begin{cases} \sum_{k=0}^{i-1} \log \rho_k, & i > 0 \\ 0, & i = 0 \\ \sum_{k=i}^{-1} -\log \rho_k, & i < 0 \end{cases}$$

Trap: An atypical section of environment where the potential is increasing.

Time to cross a trap is exponential in the height of the uphill.

Largest uphill of $V(\cdot)$ in $[0,n]$ is $\sim \frac{1}{s} \log n$ (Erdös & Renyi '70).

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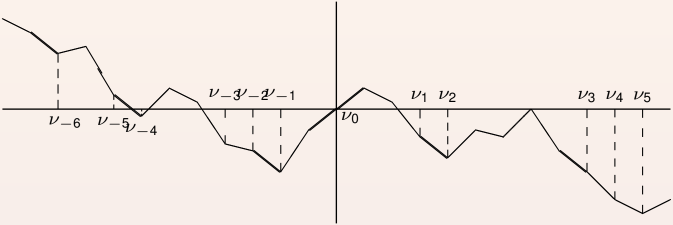
Blocks of the environment

Ladder locations $\{\nu_n\}$ defined by $\nu_0 = 0$,

$$\nu_n := \inf\{i > \nu_{n-1} : V(i) < V(\nu_{n-1})\}$$
$$\nu_{-n} := \sup\{j < \nu_{-n+1} : V(k) > V(j) \quad \forall k < j\}$$



Define a new measure on environments

$$Q(\cdot) = P\left(\cdot \,|\, \{V(i) > 0, \forall i < 0\}\right)$$

Under Q, the environment is stationary under shifts of the ν_i.

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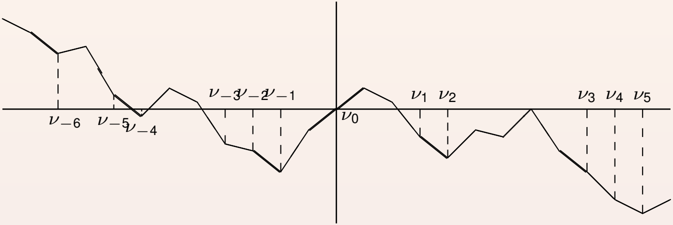

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Heuristics of Quenched Limit Laws

$$T_{\nu_n} = \sum_{i=1}^{n} (T_{\nu_i} - T_{\nu_{i-1}}) \overset{Law}{\approx} \sum_{i=1}^{n} \exp(\mu_{i,\omega})$$

where $\mu_{i,\omega} = E_\omega(T_{\nu_i} - T_{\nu_{i-1}}) \approx \sqrt{Var_\omega(T_{\nu_i} - T_{\nu_{i-1}})}$.

Quenched CLT? Only if

$$\lim_{n\to\infty} \max_{i\le n} \frac{\mu_{i,\omega}^2}{Var_\omega T_{\nu_n}} = 0, \quad P - a.s.$$

Exponential limit if

$$\lim_{n\to\infty} \max_{i\le n} \frac{\mu_{i,\omega}^2}{Var_\omega T_{\nu_n}} = 1, \quad P - a.s.$$

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Theorem (P. '07)

Assume $s < 2$*. Then* $\exists b > 0$ *s.t.*

$$\lim_{n\to\infty} Q\left(\frac{Var_\omega T_{\nu_n}}{n^{2/s}} \le x \right) = L_{\frac{s}{2},b}(x).$$

α-stable process with $\alpha < 1$ has jumps.
This hints that when $s < 2$

$$\liminf_{n\to\infty} Q\left(\max_{i\le n} \frac{\mu_{i,\omega}^2}{Var_\omega T_{\nu_n}} < \delta \right) > 0$$

and

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Contrast with the annealed results:

$$s \in (0,1) \Rightarrow \lim_{n\to\infty} \mathbb{P}\left(\frac{T_n}{n^{1/s}} \leq x \right) = L_{s,b}(x)$$

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Part II: Annealed Large Deviations for Multidimensional RWRE

Large Deviations: Definitions

Rate function: A lower semi-continuous function $h : \mathbb{R}^d \to [0, \infty]$.

Good rate function: $\{x : |h(x)| \leq C\}$ compact $\forall C < \infty$.

$\xi_n \in \mathbb{R}^d$ satisfy a **large deviation principle (LDP)** if:

$$-\inf_{x \in \Gamma^\circ} h(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P\left(\xi_n \in \Gamma\right) \leq \limsup_{n \to \infty} \frac{1}{n} \log P\left(\xi_n \in \Gamma\right) \leq -\inf_{x \in \overline{\Gamma}} h(x),$$

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$$-\inf_{x\in\Gamma^\circ} h(x) \leq \liminf_{n\to\infty} \frac{1}{n} \log P\left(\xi_n \in \Gamma\right) \\ \leq \limsup_{n\to\infty} \frac{1}{n} \log P\left(\xi_n \in \Gamma\right) \leq -\inf_{x\in\overline{\Gamma}} h(x),$$

where h is a good rate function.
That is

$$\mathbb{P}(\xi_n \approx x) \approx e^{-nh(x)}.$$

LLN for multidimensional RWRE?

No known LLN in general.
(In fact no 0-1 law for transience in a given direction).

However, the random variable $V := \lim_{n\to\infty} \frac{X_n}{n}$ exists, $\mathbb{P} - a.s.$
(Due to results of Sznitman and Zerner)

Moreover, either

1. $V =: v_P$ is $\mathbb{P} - a.s.$ constant.
2. $supp(V) = \{v_-, v_+\}$, with $v_- = cv_+$ for some $c \leq 0$.

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Annealed Large Deviations

Theorem (Varadhan '03)

Let X_n be a uniformly elliptic, nearst neighbor RWRE on $\mathbb{Z}^d$. Then, there exists a convex good rate function $H(v)$ such that $\frac{X_n}{n}$ satisfies an annealed LDP with rate function $H(v)$.

This implies

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\|X_n - nv\| < \delta) = H(v).$$

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Zero Set of the Rate Function

Drift at the origin: $d(\omega) := E_\omega X_1$.
Possible drifts: $\mathcal{K} := conv\,(supp\,(d(\omega)))$.
Nestling: $0 \in \mathcal{K}$.
Non-nestling: $0 \notin \mathcal{K}$.

Theorem (Varadhan '03)

The set $Z := \{v : H(v) = 0\}$ is either a single point or an interval containing the origin.

$$\begin{array}{lcl} \textit{Non-nestling} & \Rightarrow & Z = \{v_P\}. \\ \textit{Nestling}, supp(V) = \{v_P\} & \Rightarrow & Z = [0, v_P]. \\ \textit{Nestling}, supp(V) = \{v_-, v_+\} & \Rightarrow & Z = [v_-, v_+]. \end{array}$$

Varadhan's proof

X_n is not a Markov chain (long term memory).
Study the *comets* of the random walk:

$$W_n := (-X_n, -X_n + X_1, \dots, -X_n + X_{n-1}, 0)$$

W_n is a Markov chain (on a horrible state space $\overline{W}$).
Obtain a LDP for the empirical distribution process

$$\mathcal{R}_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{W_n}$$

with rate function $\mathcal{J}(\mu)$.
Contract for LDP for $\frac{X_n}{n}$: $H(v) = \inf_{m(\mu)=v} \mathcal{J}(\mu)$.

Properties of the Annealed Rate Function $H(v)$

Theorem (P., Zeitouni '08)

Assume the law P is non-nestling. Then, $H(v)$ is analytic in a neighborhood of v_P.

Idea:

1. Define a new function $J(v)$, which is analytic near v_P.
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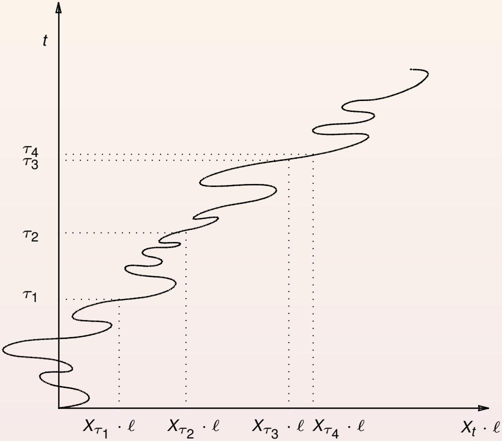
Regeneration Times

Let $\ell \in \mathbb{R}^d$ with $\|\ell\|_2 = 1$.
Regeneration times (in direction ℓ):



Regeneration Times

Assume that $\mathbb{P}(\lim_{n\to\infty} X_n \cdot \ell = +\infty) = 1$.
Define $\overline{\mathbb{P}}(\cdot) := \mathbb{P}\left(\cdot | X_n \cdot \ell \geq 0, \ \forall n\right)$.

$(X_{\tau_1}, \tau_1), (X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), (X_{\tau_3} - X_{\tau_2}, \tau_3 - \tau_2), \ldots$

- independent sequence under $\mathbb{P}$
- i.i.d. under $\overline{\mathbb{P}}$

Moreover,

$$v_P := \lim_{n\to\infty} \frac{X_n}{n} = \frac{\overline{\mathbb{E}} X_{\tau_1}}{\overline{\mathbb{E}} \tau_1}$$

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The function $\mathcal{I}$

Define for $\lambda \in \mathbb{R}^{d+1}$

$$\Lambda(\lambda) := \log \overline{\mathbb{E}} e^{\lambda \cdot (X_{\tau_1}, \tau_1)},$$

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$$\mathcal{I}(x,t) := \sup_{\lambda \in \mathbb{R}^{d+1}} \lambda \cdot (x,t) - \Lambda(\lambda).$$

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- $\mathcal{I}(x,t)$ is convex.
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Let

$$J(v) := \inf_{r\in(0,1]} r\,\mathcal{I}\left(\frac{v}{r},\frac{1}{r}\right).$$

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$$\lim_{\delta\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\|\frac{X_n}{n}-v\|<\delta\right)\leq -J(v),$$

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Sketch of the proof

Idea:

$$\overline{\mathbb{P}}(X_n \approx nv) \approx \overline{\mathbb{P}}(X_{\tau_k} \approx nv,\ \tau_k \approx n,\ k = rn)$$
$$\approx e^{-nr\mathcal{I}\left(\frac{v}{r},\frac{1}{r}\right)}$$

Lower bound:

Force $X_{\tau_k} \approx nv$ and $\tau_k \approx n$ for some $k = rn$.
Choose optimal r.

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Harder. Need to show that above strategy is optimal.
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Fix $r \in (0,1]$, and let $k = rn$.

$$\begin{aligned}
&\frac{1}{n}\log \overline{\mathbb{P}}(\|X_n - nv\| < 2\delta n) \\
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Limit as $k \to \infty$ and then $\delta \to 0$: $r\,\mathcal{I}\left(\frac{v}{r}, \frac{1}{r}\right)$.

This lower bound is true for all $r \in (0,1]$ and so

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First, note that by Chebychev's inequality

$$\overline{\mathbb{P}}(X_{\tau_k} = x,\ \tau_k = t) \le e^{-\lambda\cdot(x,t)}\overline{\mathbb{E}}e^{\lambda\cdot(X_{\tau_k},\tau_k)}$$
$$= e^{-\lambda\cdot(x,t)+k\Lambda(\lambda)} = e^{-k\left(\lambda\cdot(\frac{x}{k},\frac{t}{k})-\Lambda(\lambda)\right)}.$$

True for any $\lambda \in \mathbb{R}^{d+1}$ thus

$$\overline{\mathbb{P}}(X_{\tau_k} = x,\ \tau_k = t) \le e^{-k\mathcal{I}(\frac{x}{k},\frac{t}{k})} = e^{-t\frac{k}{t}\mathcal{I}(\frac{x}{t}\frac{t}{k},\frac{t}{k})} \le e^{-tJ(\frac{x}{t})}.$$

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Would like to say that

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Upper bound

Since P is non-nestling, τ_1 has exponential tails:

$$\overline{\mathbb{P}}(\tau_1 \geq \varepsilon n) \leq Ce^{-C\varepsilon n}.$$

Fix ε small.
Since $J(v_P) = 0$, $J(v) < C\varepsilon$ in a neighborhood of v_P.

Thus we may assume $\tau_k - \tau_{k-1} < \varepsilon n$ for all $k \leq n$.
Need an upper bound for

$$\overline{\mathbb{P}}(\exists k : \tau_k \in (n - \varepsilon n, n], \ \|X_n - nv\| < n\delta, \ \tau_{k+1} > n).$$

The event $\{\tau_k \in (n-\varepsilon n, n],\ \|X_n - nv\| < n\delta,\ \tau_{k+1} > n\}$ implies

- $\tau_k = (1-s)n$ for some $s \in [0,\varepsilon)$
- $\|X_{\tau_k} - nv\| < n(\delta + s)$
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Claim: Since $J(v)$ is quadratic near v_P, for v near v_P

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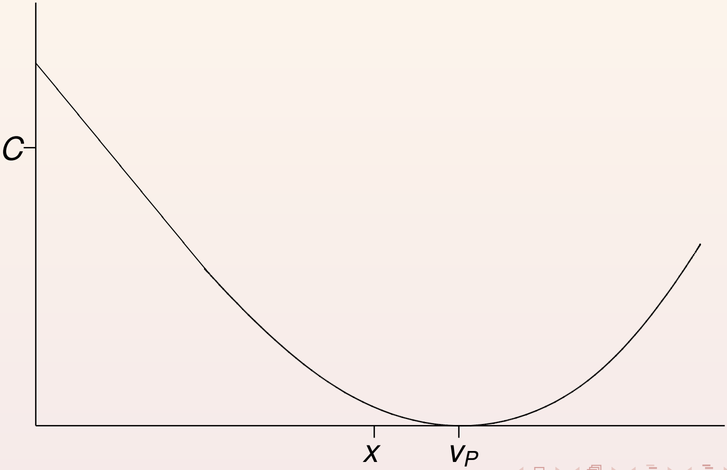

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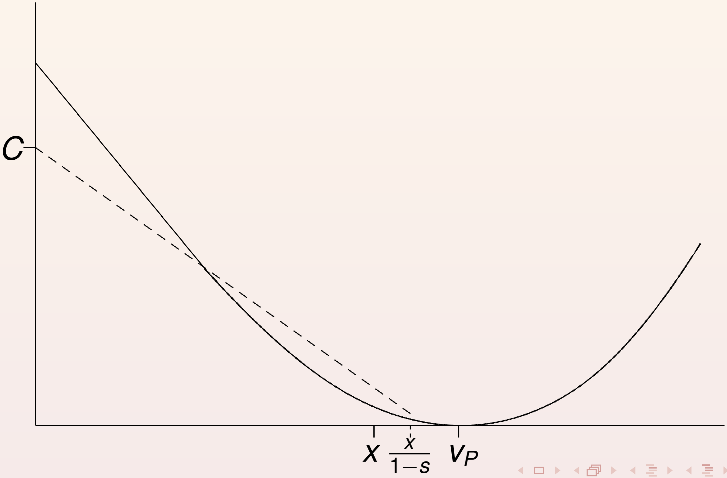

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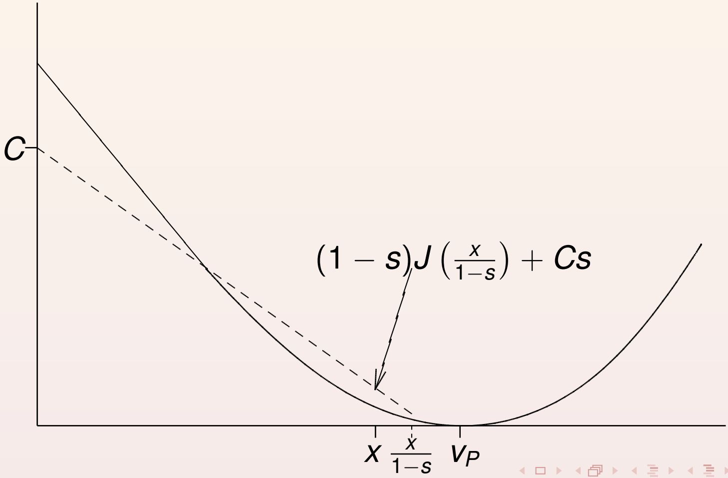

Other Results and Future Work:

- When $d = 1$, have shown $H(v) = J(v)$ for all $v > 0$ (even in nestling case).
- When $d \geq 2$, does $H(v) = J(v)$ for all $v \cdot \ell > 0$?
- Analytic behavior of $H(v)$ for ”speedup” in nestling case?
- Can anything be done for $v \cdot \ell < 0$?