Non-classical Random Walks

Simple models, surprising results, and (embarrassing) open problems

Jonathon Peterson

Department of Mathematics
University of Wisconsin

April 25, 2009

Outline

1 Classical Random Walks

Outline

1. Classical Random Walks
2. Random Walks in Random Environments

Outline

1. Classical Random Walks
2. Random Walks in Random Environments
3. Reinforced Random Walks

Outline

1. Classical Random Walks
2. Random Walks in Random Environments
3. Reinforced Random Walks
4. Excited Random Walks

Outline

1. Classical Random Walks
2. Random Walks in Random Environments
3. Reinforced Random Walks
4. Excited Random Walks

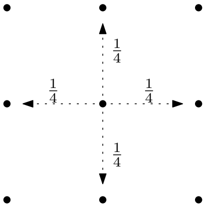
Simple Random Walks on $\mathbb{Z}^d$

A very simple model for random motion.

Transition probabilities: Probability measure μ on $\mathbb{Z}^d$.

Example:

$$\mu(x) = \begin{cases} 1/4 & |x| = 1 \\ 0 & |x| \neq 1 \end{cases}$$



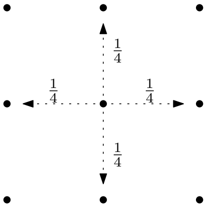
Simple Random Walks on $\mathbb{Z}^d$

A very simple model for random motion.

Transition probabilities: Probability measure μ on $\mathbb{Z}^d$.

Example:

$$\mu(x) = \begin{cases} 1/4 & |x| = 1 \\ 0 & |x| \neq 1 \end{cases}$$



Simple random walk

- Start at the origin
- Move from x to $x + e$ with probability $\mu(e)$.

Applications of Random Walks

- Modelling the Stock Market
- Discrete model for chemical diffusion
- Will an insurance company go bankrupt?

Question 1: Average Speed

X_n = location of RW after n steps.

Question 1: Average Speed

$X_n =$ location of RW after n steps.

Question: What is $\lim_{n\to\infty} \frac{X_n}{n}$? (Does limit even exist?)

Question 1: Average Speed

$X_n =$ location of RW after n steps.

Question: What is $\lim_{n\to\infty} \frac{X_n}{n}$? (Does limit even exist?)

$X_n = \xi_1 + \xi_2 + \xi_3 + \ldots + \xi_n$, where ξ_i are i.i.d. with distribution μ.

Theorem (Law of Large Numbers)

If ξ_i is a sequence of i.i.d. random variables with finite expectation, then with probability 1,

$$\lim_{n\to\infty} \frac{\xi_1 + \xi_2 + \xi_3 + \ldots + \xi_n}{n} = E[\xi_1].$$

Question 2: Recurrence/Transience

Recurrent: Infinitely many returns to the starting point.
Transient: Finitely many returns to the starting point (possibly zero).

If $E[\xi_1] = v \neq 0$, then LLN $\Rightarrow$ Transient.

Question 2: Recurrence/Transience

Recurrent: Infinitely many returns to the starting point.
Transient: Finitely many returns to the starting point (possibly zero).

If $E[\xi_1] = v \neq 0$, then LLN $\Rightarrow$ Transient.
What if $E[\xi_1] = 0$?

Question 2: Recurrence/Transience

Recurrent: Infinitely many returns to the starting point.
Transient: Finitely many returns to the starting point (possibly zero).

If $E[\xi_1] = v \neq 0$, then LLN $\Rightarrow$ Transient.
What if $E[\xi_1] = 0$?

Theorem

A simple random walk in $\mathbb{Z}^d$ *with* $E[X_1] = 0$ *is*

- *Recurrent if* $d \leq 2$.
- *Transient if* $d \geq 3$.

Question 2: Recurrence/Transience

Recurrent: Infinitely many returns to the starting point.
Transient: Finitely many returns to the starting point (possibly zero).

If $E[\xi_1] = v \neq 0$, then LLN $\Rightarrow$ Transient.
What if $E[\xi_1] = 0$?

Theorem

A simple random walk in $\mathbb{Z}^d$ with $E[X_1] = 0$ is

- *Recurrent if $d \leq 2$.*
- *Transient if $d \geq 3$.*

Math Joke:
A drunk person will eventually get home.
A drunk bird might never get home.

Question 3: Limiting Distribution

Let $v = E[\xi_1]$.
LLN $\Longrightarrow X_n \approx nv$.

Question: How far away from nv is X_n?

Question 3: Limiting Distribution

Let $v = E[\xi_1]$.
LLN $\Longrightarrow X_n \approx nv$.

Question: How far away from nv is X_n?

Theorem (Central Limit Theorem - $d = 1$)

Let $E[\xi_1] = v$ and $Var(\xi_1) = \sigma^2$. Then,

$$\lim_{n\to\infty} P\left(\frac{X_n - nv}{\sigma\sqrt{n}} \leq t\right) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

Question 3: Limiting Distribution

Let $v = E[\xi_1]$.
LLN $\implies X_n \approx nv$.

Question: How far away from nv is X_n?

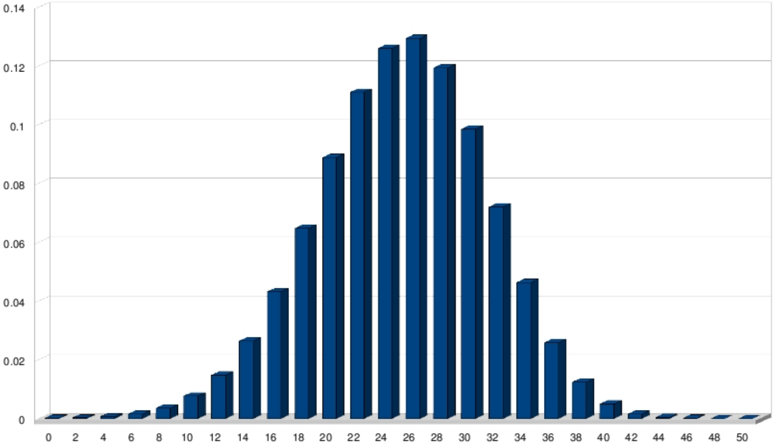
Theorem (Central Limit Theorem - $d = 1$)

Let $E[\xi_1] = v$ and $Var(\xi_1) = \sigma^2$. Then,

$$\lim_{n\to\infty} P\left(\frac{X_n - nv}{\sigma\sqrt{n}} \le t\right) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

Bar chart of probability distributions looks like a normal bell curve centered at nv.

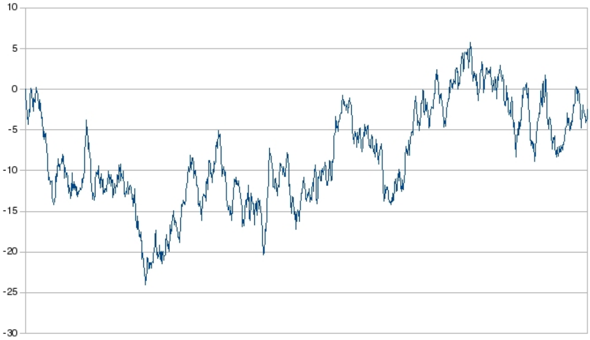
Example: Probability distribution of simple random walk with $p = .75$ after 50 steps.



Brownian Motion

Random walks are discrete approximations of continuous models.
Central Limit Theorem: Scale time by n and space by $\sqrt{n}$.

$$B_n(t) = \frac{X_{nt} - ntv}{\sigma\sqrt{n}}.$$



Outline

1 Classical Random Walks

2 Random Walks in Random Environments

3 Reinforced Random Walks

4 Excited Random Walks

RWRE on $\mathbb{Z}^d$

Environment: Transition probabilities ω_x assigned to each $x \in \mathbb{Z}^d$.

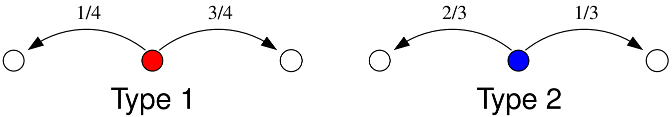
Random Environment: The ω_x are randomly assigned.

RWRE on $\mathbb{Z}^d$

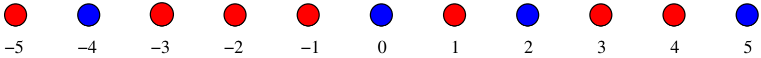
Environment: Transition probabilities ω_x assigned to each $x \in \mathbb{Z}^d$.

Random Environment: The ω_x are randomly assigned.

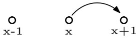
Example: (Dimension 1) Two types of environments:

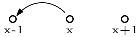


Random environment: Choose type 1 with probability p.



One-dimensional RWRE: Recurrence/Transience

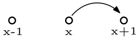
$\omega_x = P($  $)$

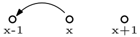
$1 - \omega_x = P($  $)$

Question 2: When is the RWRE transient to $+\infty$?

Reasonable guess: $E[\omega_0] = P(X_1 = 1) > \frac{1}{2}$.

One-dimensional RWRE: Recurrence/Transience

$\omega_x = P($ 

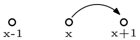
$) \qquad 1 - \omega_x = P($ 

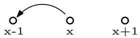
$)$

Question 2: When is the RWRE transient to $+\infty$?

Reasonable guess: $E[\omega_0] = P(X_1 = 1) > \frac{1}{2}$. WRONG

One-dimensional RWRE: Recurrence/Transience

$\omega_x = P($ 

$)$ $1 - \omega_x = P($ 

$)$

Question 2: When is the RWRE transient to $+\infty$?

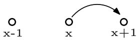
Reasonable guess: $E[\omega_0] = P(X_1 = 1) > \frac{1}{2}$. WRONG

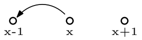
Theorem (Solomon '75)

Let $\rho_x = \frac{1-\omega_x}{\omega_x}$. *Then*

- $X_n \to +\infty \iff E[\ln(\rho_0)] < 0$
- $X_n \to -\infty \iff E[\ln(\rho_0)] > 0$
- X_n *recurrent* $\iff E[\ln(\rho_0)] = 0$.

One-dimensional RWRE: Recurrence/Transience

$\omega_x = P($ 

$)$ $\quad 1 - \omega_x = P($ 

$)$

Question 2: When is the RWRE transient to $+\infty$?

Reasonable guess: $E[\omega_0] = P(X_1 = 1) > \frac{1}{2}$. WRONG

Theorem (Solomon '75)

Let $\rho_x = \frac{1-\omega_x}{\omega_x}$. *Then*

- $X_n \to +\infty \iff E\left[\ln(\rho_0)\right] < 0$
- $X_n \to -\infty \iff E\left[\ln(\rho_0)\right] > 0$
- X_n *recurrent* $\iff E\left[\ln(\rho_0)\right] = 0.$

Follows from formula the probability of reaching $-x$ before y.

One-dimensional RWRE: LLN

Let $E[\ln(\rho_0)] < 0$ (that is $X_n \longrightarrow +\infty$).

Question 2: What is a formula for $v = \lim_{n\to\infty} \frac{X_n}{n} \longrightarrow v$?

One-dimensional RWRE: LLN

Let $E[\ln(\rho_0)] < 0$ (that is $X_n \longrightarrow +\infty$).

Question 2: What is a formula for $v = \lim_{n\to\infty} \frac{X_n}{n} \longrightarrow v$?

Reasonable guess: $v = 2E[\omega_0] - 1 = E[X_1]$.

One-dimensional RWRE: LLN

Let $E\left[\ln(\rho_0)\right] < 0$ (that is $X_n \longrightarrow +\infty$).

Question 2: What is a formula for $v = \lim_{n\to\infty} \frac{X_n}{n} \longrightarrow v$?

Reasonable guess: $v = 2E[\omega_0] - 1 = E[X_1]$. WRONG

One-dimensional RWRE: LLN

Let $E[\ln(\rho_0)] < 0$ (that is $X_n \longrightarrow +\infty$).

Question 2: What is a formula for $v = \lim_{n\to\infty} \frac{X_n}{n} \longrightarrow v$?

Reasonable guess: $v = 2E[\omega_0] - 1 = E[X_1]$. WRONG

Question 3: When is $\frac{X_n}{n} \longrightarrow v > 0$?

One-dimensional RWRE: LLN

Let $E\left[\ln(\rho_0)\right] < 0$ (that is $X_n \longrightarrow +\infty$).

Question 2: What is a formula for $v = \lim_{n\to\infty} \frac{X_n}{n} \longrightarrow v$?

Reasonable guess: $v = 2E[\omega_0] - 1 = E[X_1]$. WRONG

Question 3: When is $\frac{X_n}{n} \longrightarrow v > 0$?

Reasonable guess: Whenever $X_n \longrightarrow +\infty$.

One-dimensional RWRE: LLN

Let $E\left[\ln(\rho_0)\right] < 0$ (that is $X_n \longrightarrow +\infty$).

Question 2: What is a formula for $v = \lim_{n\to\infty} \frac{X_n}{n} \longrightarrow v$?

Reasonable guess: $v = 2E[\omega_0] - 1 = E[X_1]$. WRONG

Question 3: When is $\frac{X_n}{n} \longrightarrow v > 0$?

Reasonable guess: Whenever $X_n \longrightarrow +\infty$. WRONG

One-dimensional RWRE: LLN

Let $E\left[\ln(\rho_0)\right] < 0$ (that is $X_n \longrightarrow +\infty$).

Question 2: What is a formula for $v = \lim_{n\to\infty} \frac{X_n}{n} \longrightarrow v$?

Reasonable guess: $v = 2E[\omega_0] - 1 = E[X_1]$. WRONG

Question 3: When is $\frac{X_n}{n} \longrightarrow v > 0$?

Reasonable guess: Whenever $X_n \longrightarrow +\infty$. WRONG

Theorem (Solomon '75)

If $E_P \ln(\rho_0) < 0$, *then* $\lim_{n\to\infty} \frac{X_n}{n} = \begin{cases} \frac{1-E[\rho_0]}{1+E[\rho_0]} & E\left[\rho_0\right] < 1 \\ 0 & E\left[\rho_0\right] \geq 1. \end{cases}$

One-dimensional RWRE: LLN

Let $E\left[\ln(\rho_0)\right] < 0$ (that is $X_n \longrightarrow +\infty$).

Question 2: What is a formula for $v = \lim_{n\to\infty} \frac{X_n}{n} \longrightarrow v$?

Reasonable guess: $v = 2E[\omega_0] - 1 = E[X_1]$. WRONG

Question 3: When is $\frac{X_n}{n} \longrightarrow v > 0$?

Reasonable guess: Whenever $X_n \longrightarrow +\infty$. WRONG

Theorem (Solomon '75)

If $E_P \ln(\rho_0) < 0$, *then* $\lim_{n\to\infty} \frac{X_n}{n} = \begin{cases} \frac{1-E[\rho_0]}{1+E[\rho_0]} & E\left[\rho_0\right] < 1 \\ 0 & E\left[\rho_0\right] \geq 1. \end{cases}$

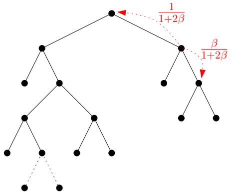
Follows from computation of $\mathbb{E}$[Time to go one unit right].

RWRE on Trees

Fix $\beta > 1$.

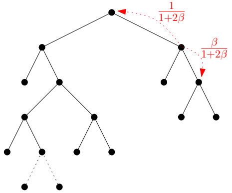
Node with k "children", RWRE moves

- up with probability $\frac{1}{1+k\beta}$
- down with probability $\frac{\beta}{1+k\beta}$.



RWRE on Trees

Fix $\beta > 1$.



Node with k “children”, RWRE moves

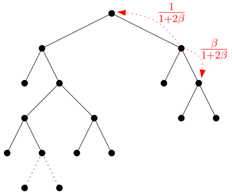
- up with probability $\frac{1}{1+k\beta}$
- down with probability $\frac{\beta}{1+k\beta}$.

Question 3: Does the RWRE move down with positive speed?

Reasonable guess: Yes. $\beta > 1$ gives bias downward.

RWRE on Trees

Fix $\beta > 1$.



Node with k "children", RWRE moves

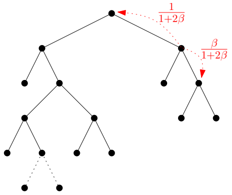
- up with probability $\frac{1}{1+k\beta}$
- down with probability $\frac{\beta}{1+k\beta}$.

Question 3: Does the RWRE move down with positive speed?

Reasonable guess: Yes. $\beta > 1$ gives bias downward. WRONG

RWRE on Trees

Fix $\beta > 1$.



Node with *k* "children", RWRE moves

- up with probability $\frac{1}{1+k\beta}$
- down with probability $\frac{\beta}{1+k\beta}$.

Question 3: Does the RWRE move down with positive speed?

Reasonable guess: Yes. $\beta > 1$ gives bias downward.   WRONG

Theorem (Lyons, Pemantle, & Peres '96)

There exists a constant β_c such that

- *Positive speed for $\beta \in (1, \beta_c)$*   *(weak bias)*
- *Zero speed for $\beta \in [\beta_c, \infty)$.*   *(strong bias)*

RWRE on $\mathbb{Z}^d$

RWRE on $\mathbb{Z}^d$ are very difficult when $d \geq 2$.

RWRE on $\mathbb{Z}^d$

RWRE on $\mathbb{Z}^d$ are very difficult when $d \geq 2$.

Embarrassing Open Problem

How do you tell if the RWRE is recurrent/transient?

RWRE on $\mathbb{Z}^d$

RWRE on $\mathbb{Z}^d$ are very difficult when $d \geq 2$.

Embarrassing Open Problem

How do you tell if the RWRE is recurrent/transient?

Embarrassing Open Problem

For $0 \neq \ell \in \mathbb{R}^d$ *is*

$$\mathbb{P}(\lim_{n\to\infty} X_n \cdot \ell = +\infty) \in \{0, 1\}?$$

That is, is there a direction of transience?

RWRE on $\mathbb{Z}^d$

RWRE on $\mathbb{Z}^d$ are very difficult when $d \geq 2$.

Embarrassing Open Problem

How do you tell if the RWRE is recurrent/transient?

Embarrassing Open Problem

For $0 \neq \ell \in \mathbb{R}^d$ *is*

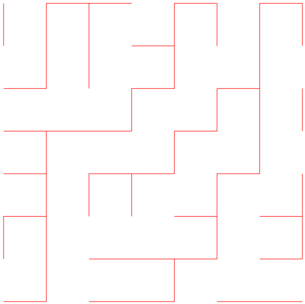
$$\mathbb{P}(\lim_{n\to\infty} X_n \cdot \ell = +\infty) \in \{0,1\}?$$

That is, is there a direction of transience?

There are examples with two possible directions for transience.

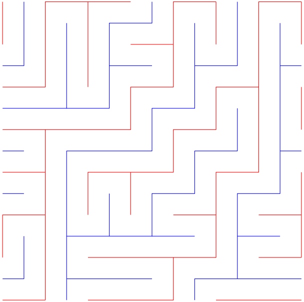
A Strange Example

Random up/right tree,



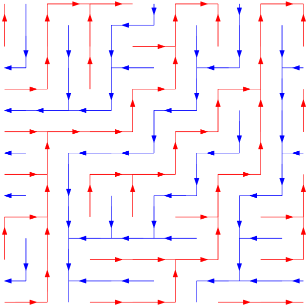
A Strange Example

Random up/right tree, and the dual down/left tree.



A Strange Example

Random up/right tree, and the dual down/left tree.



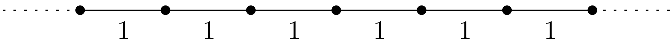
Can assign probabilities so that

$$P(\nearrow) = P(\swarrow) = \frac{1}{2}$$

Outline

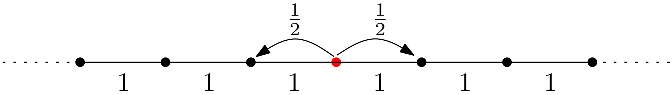
1. Classical Random Walks
2. Random Walks in Random Environments
3. Reinforced Random Walks
4. Excited Random Walks

Reinforced Random Walks



Start with all edges having weight = 1.

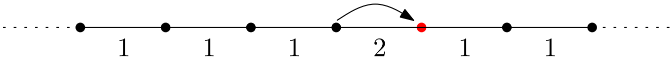
Reinforced Random Walks



Start with all edges having weight = 1.

Transition probabilities proportional to edge weights.

Reinforced Random Walks

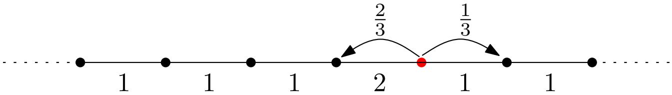


Start with all edges having weight = 1.

Transition probabilities proportional to edge weights.

When an edge is crossed, increase the weight by 1.

Reinforced Random Walks



Start with all edges having weight = 1.

Transition probabilities proportional to edge weights.

When an edge is crossed, increase the weight by 1.

Move according to updated edge weights.

ERRW as a RWRE

RWRE: Transition probabilities fixed (randomly) at start.
ERRW: Transition probabilities change over time.

ERRW as a RWRE

RWRE: Transition probabilities fixed (randomly) at start.
ERRW: Transition probabilities change over time.

Theorem (Diaconis '88)

An ERRW has the same distribution as a certain RWRE.

- Simple random environment if graph is a tree (even infinite).

ERRW as a RWRE

RWRE: Transition probabilities fixed (randomly) at start.
ERRW: Transition probabilities change over time.

Theorem (Diaconis '88)

An ERRW has the same distribution as a certain RWRE.

- Simple random environment if graph is a tree (even infinite).
- Example: ERRW on $\mathbb{Z}$

$$\omega_x \sim \begin{cases} Beta(2,1) & x \leq -1 \\ U(0,1) & x = 0 \\ Beta(1,2) & x \geq 1 \end{cases}$$

ERRW as a RWRE

RWRE: Transition probabilities fixed (randomly) at start.
ERRW: Transition probabilities change over time.

Theorem (Diaconis '88)

An ERRW has the same distribution as a certain RWRE.

- Simple random environment if graph is a tree (even infinite).
- Example: ERRW on $\mathbb{Z}$

$$\omega_x \sim \begin{cases} Beta(2,1) & x \leq -1 \\ U(0,1) & x = 0 \\ Beta(1,2) & x \geq 1 \end{cases}$$

- Complicated random environment if graph has cycles.

Recurrence of ERRW

Reinforcement should make recurrence more likely.

Recurrence of ERRW

Reinforcement should make recurrence more likely.

Embarrassing Open Problem

Is an ERRW on $\mathbb{Z}^d$ with $d \geq 2$ recurrent?

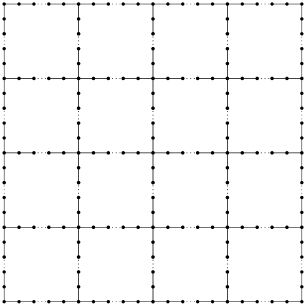
Recurrence of ERRW

Reinforcement should make recurrence more likely.

Embarrassing Open Problem

Is an ERRW on $\mathbb{Z}^d$ with $d \geq 2$ recurrent?

Partial progress when $d = 2$ (Merkel & Rolles '08).



Outline

1 Classical Random Walks

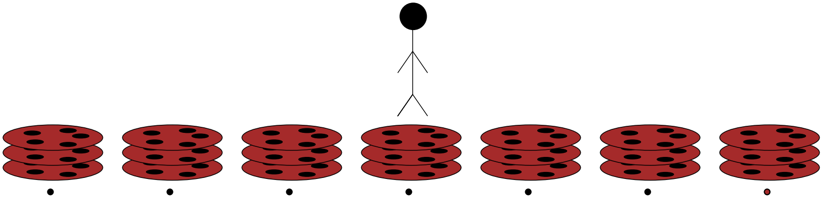
2 Random Walks in Random Environments

3 Reinforced Random Walks

4 Excited Random Walks

Excited (Cookie) Random Walks

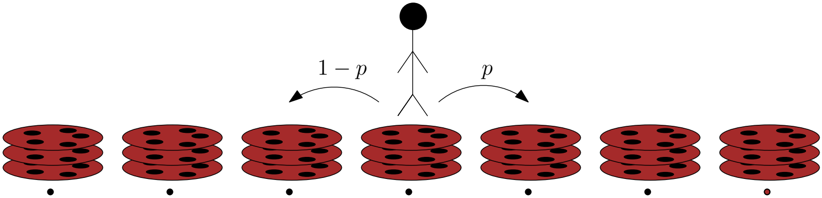
(M, p) **Cookie Random Walk**
Initially M cookies at each site.



Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.

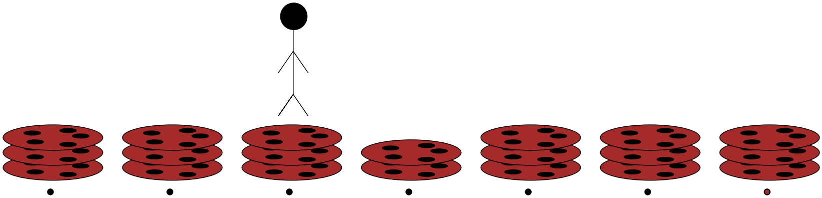
- **Cookie available:** Eat cookie. Move right with probability $p > \frac{1}{2}$



Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.

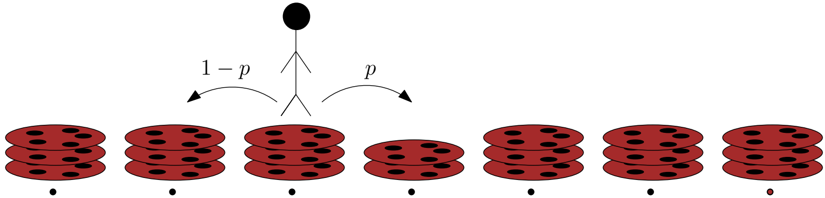
- **Cookie available:** Eat cookie. Move right with probability $p > \frac{1}{2}$



Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.

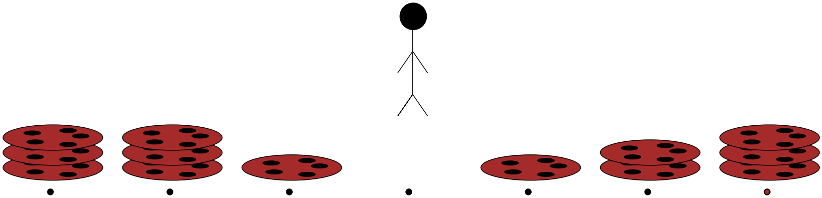
- **Cookie available:** Eat cookie. Move right with probability $p > \frac{1}{2}$



Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.

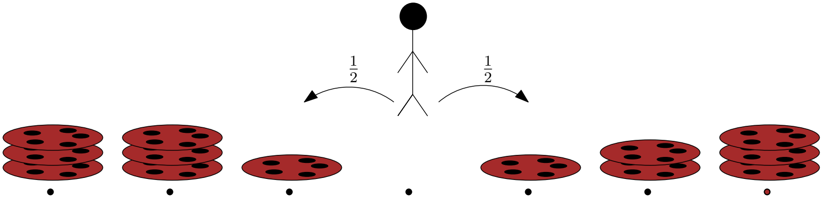
- **Cookie available:** Eat cookie. Move right with probability $p > \frac{1}{2}$
- **No cookies:** Move right/left with probability $\frac{1}{2}$.



Excited (Cookie) Random Walks

(M, p) **Cookie Random Walk**
Initially M cookies at each site.

- **Cookie available:** Eat cookie. Move right with probability $p > \frac{1}{2}$
- **No cookies:** Move right/left with probability $\frac{1}{2}$.



Recurrence/Transience and LLN

M cookies with strength $p_1, p_2, \ldots p_M > \frac{1}{2}$.

$$\alpha = \sum_{i=1}^{M} (2p_i - 1) - 1.$$

Theorem (Zerner '05)

The cookie RW is transient if and only if $\alpha > 0$.

Recurrence/Transience and LLN

M cookies with strength $p_1, p_2, \dots p_M > \frac{1}{2}$.

$$\alpha = \sum_{i=1}^{M} (2p_i - 1) - 1.$$

Theorem (Zerner '05)

The cookie RW is transient if and only if $\alpha > 0$.

Theorem (Basdevant & Singh '08)

Let $v = \lim_{n\to\infty} X_n/n = v$. Then, $v > 0 \iff \alpha > 1$.

(M, p) cookie random walk

M cookies all of strength $p > \frac{1}{2}$. $\quad v > 0 \iff p > \frac{1}{2} + \frac{1}{M}$.

(M, p) cookie random walk

M cookies all of strength $p > \frac{1}{2}$. $v > 0 \iff p > \frac{1}{2} + \frac{1}{M}$.

Embarrassing Open Problem

Find a fomula for $v = v(M, p)$.

(M, p) cookie random walk

M cookies all of strength $p > \frac{1}{2}$. $\quad v > 0 \iff p > \frac{1}{2} + \frac{1}{M}$.

Embarrassing Open Problem

Find a fomula for $v = v(M, p)$.

Figure: Simulations of speed for a $(3, p)$ cookie walk (Basdevant and Singh '08)