Project Based Math 112, Fall 2001
Activity §8.1 — Introduction to Infinite Sequences

1 Introduction

An infinite sequence (or just "sequence") is an infinite listing of numbers, like

$1, 2, 3, 4, 5, 6, \ldots$, [the integers]
$1, 1/2, 1/3, 1/4, \ldots$, [reciprocals of the integers]
$0, 0, 0, 0, 0, \ldots$, [all zeros]
$8.17, 1, 2, 3, 3, 3, 3, 3, \ldots$ [some random stuff then all three's],
$3, 1, 4, 1, 5, 9, 2, \ldots$ [the digits of pi].

1. Create your own infinite sequence:

2. Create another:

Some notation: Just as we sometimes like to use constants like c and K instead of actual numbers like 3 and 8.64, there is a corresponding notation for sequences:

$$a_1, a_2, a_3, \ldots \quad \text{or} \quad \{a_1, a_2, a_3, \ldots\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty} \quad \text{or just} \quad \{a_n\}$$

Here n is just an index or "dummy variable". The sequence $\{a_n\}$ is the same as the sequence $\{a_k\}$. For example, the sequence $\{1/n\}$ is the same as the sequence $\{1/k\}$, since both of these represent (equal):

$$1, 1/2, 1/3, 1/4, \ldots .$$

2 Part A

Definition: If we can make the terms a_n as close to some number L as we like by taking n very large, then we say the sequence $\{a_n\}$ tends or converges to L as n tends to ∞, and write

$$a_n \to L \text{ as } n \to \infty \qquad \text{or} \qquad \lim_{n\to\infty} a_n = L.$$

In this case, we say the sequence is *convergent*, or *converges* (to L). Otherwise, we say the sequence *diverges*.

For example, the limit of the constant sequence $c, c, c, c, \ldots$ is c.

Given a sequence $\{a_n\}$, there are several possibilities as to its limit behavior:

A. The sequence $\{a_n\}$ may converge to a number L.

B. The sequence may diverge in one of three ways:

   i. The sequence may tend to $+\infty$.
   ii. The sequence may tend to $-\infty$.
   iii. The limit might not exist at all. For example, it might oscillate, like in $\{cos(n)\}$.

3. Which type is the sequence $1/k$?

4. What about $(-1)^n/n$?

5. Or $(-1)^k$?

Now write two (or more) sequences whose convergence is of each of the types listed above (you may use some of the sequences listed on the first page).

6. Sequences of type A:

7. Sequences of type Bi:

8. Sequences of type Bii:

9. Sequences of type Biii:

3 Part B

Another definition A sequence $\{a_n\}$ can also be thought of as a function f whose domain is the integers $\{1, 2, 3, \dots\}$ and whose range is the real numbers: $f(n) = 1/n$ for example.

10. The graph of such a function is an infinite sequence of points. Draw one set of axes below, and put in the graphs of the three infinite sequences (in different colors or dot styles) given by $f(n) = 1/n$, $g(n) = n$, and $h(n) = 2 - (1/n)$. Be sure to label the units on the axes.

11. Speaking informally but being as precise as you can, describe how the graphs above relate to convergence. What visually/geometrically is key to convergence?

12. Now graph $f(n) = (-1)^n/n$ and $g(n) = (-1)^n$ on the same graph below:

13. What do these graphs tell you about convergence? Why? Be as mathematically precise and convincing as you can.

Theorem: *If $f(x) \to L$ as $x \to \infty$, and $a_n = f(n)$ for every integer n, then $a_n \to L$ as $n \to \infty$.*

Thus, since $1/x \to 0$ as $x \to \infty$, then $1/n \to 0$ as $n \to \infty$. Of course, not every sequence a_n will be just $f(n)$ for an obvious formula/function $f(x)$. (For example, let a_n be the population of the world in the year n.)

14. For each of the following functions, determine using the theorem above, whether the sequence defined by the function converges.

a. $f(n) = n/(n+1)$

b. $g(k) = k\sin(1/k)$

4 Part C

Theorem: [Limit laws for convergent sequences] If $a_n \to L$ and $b_n \to M$, then

- $ca_n \to cL$,
- $a_n + b_n \to L + M$,
- $a_n b_n \to LM$,
- if M is not zero, then $a_n/b_n \to L/M$,
- $a_n^p \to L^p$ if $p > 0$ and $a_n > 0$, for all n.

15. Use limits you found previously in this activity, and the limit laws, to find the limits of the following sequences:

a. $f(n) = 2n/(n+1)$

b. $g(n) = \dfrac{n}{(n+1)} + \dfrac{2}{n}$