Math 428

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SHOW ALL OF YOUR WORK. You may only discuss this exam with the lecturer or the TA. Using sources other than the textbook is allowed and encouraged but please include references to any books you use.

1. Suppose a particle moves in \mathbb{R}^3 . Its position at time t is denoted by $\mathbf{x}(t) = (x(t), y(t), z(t))$. Let $T = \frac{m}{2}\dot{\mathbf{x}}^2$ be the kinetic energy of the particle and $V = V(\mathbf{x})$ be its potential energy. The principle of least action states that the particle will move from time t_1 to t_2 from the point \mathbf{x}_1 to the point \mathbf{x}_2 in such a way as to minimize the action integral

$$A = \int_{t_1}^{t_2} T - V \, dt.$$

Show that the Euler-Lagrange equations for this problem are Newton's equations of motion.

2. A wave guide is a long, hollow tube that is used to confine electromagnetic (or acoustic) waves with the purpose of propagate them with minimal decay. Assume that we have an infinite wave guide $(-\infty \le x \le \infty)$ with rectangular cross section of width ℓ ($0 \le y \le \ell$) and height $h(0 \le z \le h)$. Assume that the electromagnetic waves inside the guide are modeled by the wave equation with speed of light c and that the tube shell is grounded so that we have Dirichlet boundary conditions. By using separation of variables show that the potential inside the wave guide at time t and position $\mathbf{x} = (x, y, z)$ is given by

$$u(\mathbf{x},t) = \int_{-\infty}^{\infty} \sum_{n,m=1}^{\infty} \hat{f}_{n,m}(k) e^{i(kx-\omega t)} \sin[n\pi y/\ell] \, \sin[m\pi z/h] \, \frac{dk}{2\pi}$$

where $\omega^2 = c^2 [k^2 + (n\pi y/\ell)^2 + (m\pi z/h)^2]$. \hat{f} is the Fourier transform of the initial condition in the variable x of the Fourier coefficients in y and z, that is

$$\hat{f}_{m,n}(k) = \frac{4}{h\ell} \int_0^h \int_0^\ell \int_{-\infty}^\infty \sin[n\pi y/\ell] \, \sin[m\pi z/h] e^{-ikx} u(x,y,z,0) \, dx \, dy \, dz.$$

3. Use the method of reflection to find the Green's function of Δ for the strip $-\infty < x < \infty$, 0 < y < a.

- 4. Solve problems 21 and 22 of section 7.4 of the textbook.
- 5. Show that the solution to the heat equation

$$u_t = \kappa u_{xx} \quad t > 0, -\infty < x < \infty$$

with initial condition u(x,0) = f(x) can be written in terms of the Fourier transform \hat{f} of the initial condition as

$$u(x,t) = \int_{-\infty}^{\infty} e^{ikx} e^{-\kappa tk^2} \hat{f}(k) \, dk.$$

Investigate what Laplace's asymptotic method is. Use it to deduce that the temperature distribution is of the form

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} f(\xi) \, d\xi + \mathcal{O}\left(\frac{1}{t}\right)$$

when t is large.