DYNAMICS OF A SYSTEM OF TWO COUPLED MEMS OSCILLATORS

Richard H. Rand[1,2], Alan T. Zehnder[1], Bhattacharjee Shayak[1]
[1] *Theoretical and Applied Mechanics, Sibley School of Mechanical and Aerospace Engineering, Cornell University*
[2] *Department of Mathematics, Cornell University, Ithaca, NY, USA*

Summary: A model of a simplified MEMS device which has been previously shown to support a limit cycle [1] is used to model a pair of coupled MEMS oscillators. The stability and bifurcation of in-phase and out-of-phase modes is investigated.

1 Introduction

This work is concerned with a type of MEMS device in which a laser is used to measure the device's deformation via an interference pattern. As a side effect, the laser heats the device and affects the interference gap, resulting in a feedback loop which causes the device to vibrate in a limit cycle. A model of this process has been given in [1] and [2]:

$$z'' + \frac{1}{Q}(z' - DT') + (1 + CT)(z - DT) + \beta(z - DT)^3 = 0 \quad \text{and} \quad T' + BT = AP(\alpha + \gamma \sin^2 2\pi(z - z_0)). \quad (1)$$

Here z is the displacement of a mechanical oscillator and T is its temperature due to laser illumination. In the mechanical equation Q is the quality factor, C is the stiffness change due to temperature, D is the displacement due to temperature and β is the coefficient of the cubic nonlinearity. In the thermal equation the quantities α and γ represent the average and contrast of the absorption of laser power, P is the laser power, A and B represent the thermal mass and heat loss rate. The offset z_0 models the equilibrium position of the oscillator with respect to the interference field created by the oscillator/gap/substrate stack. This sophisticated model, which includes effects of damping, stiffness change due to heating, periodic dependence of light absorption on interferometric gap, and nonlinearity, was shown in [2] to support limit cycle oscillations.

In the present work and in [1], a simplified model of the foregoing equation is considered which omits damping and various other effects:

$$z'' + z = T, \qquad T' + T = z^2 - pz. \quad (2)$$

where the parameter p stands for z_0 in Eq.(1) and can be given a representative value of 0.1. Using perturbation methods, it was shown in [1] that this system admits a limit cycle having the approximate form:

$$z = A\cos\omega t + A^2\left(\frac{1}{2} - \frac{1}{15}\sin 2\omega t - \frac{1}{30}\cos 2\omega t\right), \qquad T = A^2\left(\frac{1}{2} + \frac{1}{5}\sin 2\omega t + \frac{1}{10}\cos 2\omega t\right). \quad (3)$$

where

$$A = \frac{\sqrt{10p}}{3} \quad \text{and} \quad \omega = 1 - \frac{p}{27}. \quad (4)$$

In the present work we use Eq.(2) to model a system of two coupled MEMS oscillators:

$$z_1'' + z_1 = T_1 + \alpha(z_2 - z_1), \qquad T_1' + T_1 = z_1^2 - pz_1, \quad (5)$$

$$z_2'' + z_2 = T_2 + \alpha(z_1 - z_2), \qquad T_2' + T_2 = z_2^2 - pz_2. \quad (6)$$

where α is a coupling constant. Here we have assumed that the two MEMS oscillators are linked mechanically, but that they are thermally isolated from each other.

2 Analysis

Inspection of Eqs.(5),(6) shows that they exhibit an in-phase (IP) mode, i.e. the 6-dimensional system (5),(6) admits a 3-dimensional invariant manifold $z_1 = z_2, T_1 = T_2$. The flow on this IP manifold is the same as the flow (3),(4) of the individual oscillator (2). In order to study the stability of the IP mode, we recast the system (5),(6) in terms of sum and difference variables

$$x = z_1 + z_2, \quad u = z_1 - z_2, \quad y = T_1 + T_2, \quad v = T_1 - T_2. \quad (7)$$

The system (5),(6) becomes:

$$x'' + x = y, \qquad y' + y = \frac{x^2 + u^2}{2} - px, \quad (8)$$

$$u'' + (1 + 2\alpha)u = v, \qquad v' + v = xu - pu. \quad (9)$$

Here the IP manifold corresponds to the exact solution $u = v = 0$. The flow on the IP manifold can be written $x = 2F(t), y = 2G(t)$ where $F(t)$ and $G(t)$ are given by z and T in Eqs.(3). To determine stability of the IP mode, we write

$$x = 2F(t) + X, \quad y = 2G(t) + Y, \quad u = 0 + U, \quad v = 0 + V. \tag{10}$$

where X, Y, U, V are small variations. The linearized variational equations become:

$$X'' + X = Y, \qquad Y' + Y = 2F(t)X - pX, \tag{11}$$
$$U'' + (1 + 2\alpha)U = V, \qquad V' + V = 2F(t)U - pU. \tag{12}$$

Eqs.(11) represent the stability of the IP mode due to variations in the invariant $x - y$ manifold. As is well known [3], the limit cycle motion in this case is Liapunov unstable but is orbitally stable. Eqs.(12) represent the stability of the invariant manifold due to variations in directions normal to the IP invariant manifold. Differentiating Eq.(12.1) and then substituting (12.2), we obtain the following third order ODE:

$$U''' + U'' + (1 + 2\alpha)U' + (1 + 2\alpha + p + 2F(t))U = 0. \tag{13}$$

Approximating $F(t) \approx A\cos\omega t$, cf. Eqs.(3),(4), and treating p as a higher order quantity, we obtain an equation which we refer to as a third order Mathieu equation:

$$U''' + U'' + \delta\, U' + (\delta + \varepsilon\cos t)\, U = 0, \tag{14}$$

where $\delta = 1 + 2\alpha$ and $\omega \approx 1$. Using regular perturbations, we obtain expressions for the transition curves in Eq.(14). For $p = 0.1$, the analysis gives that there is a transition from stable to unstable when α is lowered beyond 0.04, a fact which is confirmed by numerical simulations.

What about an out-of-phase (OP) mode? Inspection of Eqs.(8),(9) shows that the expected $x = y = 0$ is not an exact solution. Nevertheless, numerical simulation of Eqs.(5),(6) shows that they support an OP mode. To study this mode, we introduce a small parameter ε and use Lindstedt's perturbation method. The resulting approximation for the OP mode is then inserted into the associated variational equation which yields the value of $\alpha = 0.88$ above which the OP mode loses stability, which is in good agreement with the numerically observed value of $\alpha = 0.83$.

Thus the IP mode is stable when $\alpha > 0.04$ and the OP mode is stable when $\alpha < 0.83$, which is to say that both the IP and OP modes are stable when α lies between 0.04 and 0.83. With both modes stable, we expect an unstable motion which has a stable manifold which lies on the boundary of the basins of attraction of the stable modes. Although this separatrix motion is unstable, we may seek it by numerically varying initial conditions, trying to find the point where the motion switches between the IP and OP modes. The resulting separatrix motion appears to be quasiperiodic. This situation is reminiscent of the dynamics of two coupled van der Pol oscillators [4], where a perturbation approximation for the separatrix motion was obtained.

3 Acknowledgement

This material is based upon work supported by the National Science Foundation under grant number CMMI−1634664.

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