Harmonic Functions and the Spectrum of the Laplacian on the Sierpinski Carpet

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Construction of SC

The Sierpinski Carpet, SC, is constructed by eight contraction mappings. The maps contract the unit square by a factor of 1/3 and translate to one of the eight points along the boundary. SC is the unique nonempty compact set satisfying the self-similar identity

$$\text{SC} = \bigcup_{i=0}^{7} F_i(\text{SC}).$$
Constructing the Laplacian

The Laplacian $\Delta$ on $SC$ was independently constructed by Barlow & Bass (1989) and Kusuoka & Zhou (1992). In 2009, Barlow, Bass, Kumagai, & Teplyaev showed that both methods construct the same unique Laplacian on $SC$.

We will be following Kusuoka & Zhou’s approach in which we consider average values of a function on any level $m$-cell.

We approximate the Laplacian on the carpet by calculating the graph Laplacian on the approximation graphs where vertices of the graph are cells of level $m$: 
Constructing the Laplacian

\[ \Delta_m u(x) = \sum_{y \sim x}^m (u(y) - u(x)). \]

For example the graph Laplacian of interior cell \( a \) is
\[ \Delta_m u(a) = -3u(a) + u(b) + u(c) + u(d). \]
For boundary cells, we include its neighboring virtual cells.
\[ \text{eg: } \Delta_m u(x) = -4u(x) + u(y) + u(z) + u(\tilde{x}) + u(\tilde{x}'). \]
The Laplacian on the whole carpet is the limit of the approximating graph Laplacians

$$\Delta = \lim_{m \to \infty} r^{-m} \Delta_m$$

where $r$ is the renormalization constant $r = (8\rho)^{-1}$.

So far, $\rho$ has only been determined experimentally. $\rho \approx 1.251$ and therefore $1/r \approx 10.011$. 
A harmonic function, $h$, minimizes the graph energy given a function defined along the boundary as well as satisfying $\Delta h(x) = 0$ for all interior cells $x$.

The boundary of $SC$ is defined to be the unit square containing all of $SC$. Example: Set three edges of the boundary of $SC$ to 0 and assign $\sin \pi x$ along the remaining edge and extend harmonically.

\[ \sin \pi x \]
More Harmonic Functions

\[ \sin 2\pi x \]

\[ \sin 3\pi x \]
We also wish to solve the eigenvalue problem on the Sierpinski Carpet:

$$-\Delta u = \lambda u$$

We have two types of boundary value problems:

**Neumann**

$$\partial_n u |_{\partial SC} = 0$$

Corresponds to even reflections about boundary.

ie: $$\tilde{x} = x$$

**Dirichlet**

$$u |_{\partial SC} = 0$$

Corresponds to odd reflections about the boundary.

ie: $$\tilde{x} = -x$$
\[ \Delta_m u(x) = \sum_{y \sim x}^m (u(y) - u(x)) \]

Therefore the Laplacian operator is determined by \(8^m\) linear equations.

This can be represented in an \(8^m\) square matrix.

The matrix is created in MATLAB and the eigenvalues and eigenfunctions are calculated using the built-in \textit{eigs} function.
Some eigenfunctions

Neumann: $\partial_n u|_{\partial SC} = 0$

Dirichlet: $u|_{\partial SC} = 0$
Refinement

- On level \( m + 1 \) we expect to see all \( 8^m \) eigenfunctions from level \( m \) but refined.
- The eigenvalue is renormalized by \( r = 10.011 \).

Figure: \( \phi_{5}^{(4)} \) and \( \phi_{5}^{(5)} \) with respective eigenvalues \( \lambda_{5}^{(4)} = 0.00328 \) and \( \lambda_{5}^{(4)} = 0.000328 \).
Miniaturization

Any level $m$ eigenfunction and eigenvalue miniaturizes on the level $m + 1$ carpet. It will consist of 8 copies of $\phi^{(4)}$ or $-\phi^{(4)}$.

Figure: $\phi_4^{(4)}$ and $\phi_{20}^{(5)}$ with respective eigenvalues $\lambda_4^{(4)} = \lambda_{20}^{(5)} = 0.00177$. 
Describing the eigenvalue data

Eigenvalue counting function: \( N(t) = \#\{\lambda : \lambda \leq t\} \)

- \( N(t) \) is the number of eigenvalues less than or equal to \( t \). Describes the spectrum of eigenvalues.
- We expect the \( N(t) \) to asymptotically grow like \( t^\alpha \) as \( t \to \infty \) where \( \alpha = \log 8 / \log 10.011 \approx 0.9026 \).
$N(t) = \#\{\lambda : \lambda \leq t\}$
Weyl Ratio: \( W(t) = \frac{N(t)}{t^\alpha} \quad \alpha \approx 0.9 \)
By the min-max property, we can say that $\lambda_j^{(N)} \leq \lambda_j^{(D)}$ for each $j$. Therefore, $N^{(D)}(t) \leq N^{(N)}(t)$.

What is the growth rate of $N^{(N)}(t) - N^{(D)}(t)$. We suspect there is some power $\beta$ such that $N^{(N)}(t) - N^{(D)}(t) \sim t^\beta$.

$$\beta = \frac{\log 3}{\log 10.011} \approx 0.4769$$
$N^{(N)}(t) - N^{(D)}(t)$
A stronger periodicity is apparent here.
We can eliminate the boundary of $SC$ by gluing its boundary in specific orientations. We examined three types of $SC$ fractafolds:

- Torus
- Klein Bottle
- Projective Space
Some Eigenfunctions for the Fractafoolds
How to define the normal derivative on the boundary of SC

We wish to define $\partial_n u$ on $\partial SC$ so that the Gauss-Green formula holds:

$$\mathcal{E}(u, v) = -\int_{SC} (\Delta u)v \, d\mu + \int_{\partial SC} (\partial_n u)v \, d\mu'.$$

We know that

$$\mathcal{E}_m(u, v) = \frac{1}{\rho^m} \sum_{x \sim y} (u(x) - u(y))(v(x) - v(y))$$

and

$$-\Delta_m u(x) = \frac{8^m}{\rho^m} \sum_{x \sim y} (u(x) - u(y)).$$
Sketch of how $\partial_n u$ is defined

Let $x$ be a point on $\partial SC$ and $x_m$ be the $m$-cell containing $x$. We can use the equations from the previous slide to find $\partial_m u$ remembering to give special treatment to cells on the boundary of $SC$ because we must incorporate their virtual cells.

After much rearrangement we obtain

$$\int_{\partial SC} v \partial_n u \, dx = \frac{2 \cdot 3^m}{\rho^m} \sum_{x_m \sim \partial SC} v(x_m)(f(x) - u(x)) \frac{1}{3^m}$$

which lets us define the normal derivative as:

$$\partial_n u(x) = \lim_{m \to \infty} \frac{2 \cdot 3^m}{\rho^m} (u(x) - u(x_m)).$$

The normal derivative most likely only exists as a measure.
Rates of Convergence

Let us return to the harmonic extension of $SC$ with $\sin \pi nx$ defined along one edge of the boundary. We wish to describe how the values of the cells along the bottom edge of the boundary approach 0.

For every cell, $x_m$, along the boundary we take the value of its two parent cells, $x_{m-1}$ and $x_{m-2}$. They decay exponentially. We find that exponential rate for each of the $3^m$ cells along the boundary.
Calculated Decay Rates
More data is available (and more to come) on
www.math.cornell.edu/~reu/sierpinski-carpet
including:

- List of eigenvalues and pictures of eigenfunctions for both Dirichlet and Neumann boundary value problems.
- Eigenvalue counting function data & Weyl ratios.
- Eigenvalue data on fractaafolds and covering spaces of SC.
- Trace of the Heat Kernel data.
- Dirichlet and Poisson kernel data.
- All MATLAB scripts used.