Harmonic Functions and the Spectrum of the Laplacian on the Sierpinski Carpet

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October 4, 2011
Norbert Wiener Center Seminar
Construction of the Sierpinski Carpet

To construct the Sierpinski Carpet, we start with the unit square in Euclidean space. It does not matter where the origin is in our construction. Identify the eight points \( \{q_i\}_{i=0}^7 \) as labeled below.
We then introduce eight contraction mappings, \( \{F_i\}_{i=0}^{7} \) defined by

\[
F_i(x) = \frac{1}{3}(x - q_i) + q_i.
\]

Each map contracts the unit square by a factor of 1/3 and translates the result to one of the eight points along the boundary.
Construction of SC

We now have eight smaller squares that are identical to rescaled versions of unit square that we started with. We can apply the eight contraction mappings to each of these squares.
Construction of $SC$

We can continue this iteration process indefinitely. This gives us the entire Sierpinski carpet.

In fact, $SC$ is the unique nonempty compact set satisfying the self-similar identity

$$SC = \bigcup_{i=0}^{7} F_i(SC).$$
Graph approximations

- Since it is impossible for us to apply these maps indefinitely, we usually work with a graph approximation of the fractal. For example, $SC_m$ is the graph of applying the contraction maps $m$ times.

- When looking to describe properties on $SC$ we find the property on $SC_m$ and take the limit as $m \to \infty$. 
We can categorize most self-similar fractals as either PCF (post-critically finite) or non-PCF.

For PCF fractals there is a simple construction of the Laplacian. (For Sierpinski Gasket there is a theory of spectral decimation that allows the exact computation of eigenvalues and eigenfunctions of the Laplacian).

For non-PCF fractals, we cannot apply the same theoretical tools and the experimental approach becomes increasingly important.
Constructing the Laplacian

The Laplacian $\Delta$ on $SC$ was independently constructed by Barlow & Bass (1989) and Kusuoka & Zhou (1992). In 2009, Barlow, Bass, Kumagai, & Teplyaev showed that both methods construct the same unique Laplacian on $SC$.

We will be following Kusuoka & Zhou’s approach in which we consider average values of a function on any level $m$-cell.

We approximate the Laplacian on the carpet by calculating the graph Laplacian on the approximation graphs where vertices of the graph are cells of level $m$: 
Constructing the Laplacian

\[ \Delta_m u(x) = \sum_{y \sim x}^m (u(y) - u(x)). \]

For example the graph Laplacian of interior cell \( a \) is
\[ \Delta_m u(a) = -3u(a) + u(b) + u(c) + u(d). \]

For boundary cells, we include its neighboring virtual cells.

eg: \[ \Delta_m u(x) = -4u(x) + u(y) + u(z) + u(\tilde{x}) + u(\tilde{x}'). \]
Construction of the Laplacian

The Laplacian on the whole carpet is the limit of the approximating graph Laplacians

$$\Delta = \lim_{m \to \infty} \rho^{-m} \Delta_m$$

where $\rho$ is the renormalization constant $\rho = (8r)^{-1}$.

So far, $r$ has only been determined experimentally. $r \approx 1.251$ and therefore $1/\rho \approx 10.011$. 
A harmonic function, $h$, minimizes the graph energy given a function defined along the boundary as well as satisfying $\Delta h(x) = 0$ for all interior cells $x$.

The boundary of $SC$ is defined to be the unit square containing all of $SC$. Example: Set three edges of the boundary of $SC$ to 0 and assign $\sin \pi x$ along the remaining edge and extend harmonically.

$$\sin \pi x$$
More Harmonic Functions

\[ \sin 2\pi x \]

\[ \sin 3\pi x \]
Boundary Value Problems

We also wish to solve the eigenvalue problem on the Sierpinski Carpet:

$$-\Delta u = \lambda u$$

We have two types of boundary value problems:

**Neumann**

$$\partial_n u \mid_{\partial SC} = 0$$

Corresponds to even reflections about boundary.

ie: \( \tilde{x} = x \)

**Dirichlet**

$$u \mid_{\partial SC} = 0$$

Corresponds to odd reflections about the boundary.

ie: \( \tilde{x} = -x \)
Some eigenfunctions

Neumann: \( \partial_n u |_{\partial SC} = 0 \)

Dirichlet: \( u |_{\partial SC} = 0 \)
More Neumann eigenfunctions: N-2
More Neumann eigenfunctions: N-3
More Neumann eigenfunctions: N-4
More Neumann eigenfunctions: N-5
More Neumann eigenfunctions: N-6
More Neumann eigenfunctions: N-7
More Neumann eigenfunctions: N-8
More Neumann eigenfunctions: N-9
More Neumann eigenfunctions: N-10
More Neumann eigenfunctions: N-19
More Neumann eigenfunctions: N-42
More Dirichlet eigenfunctions: D-2
More Dirichlet eigenfunctions: D-3
More Dirichlet eigenfunctions: D-4
More Dirichlet eigenfunctions: D-5
More Dirichlet eigenfunctions: D-6
More Dirichlet eigenfunctions: D-7
More Dirichlet eigenfunctions: D-8
More Dirichlet eigenfunctions: D-9
More Dirichlet eigenfunctions: D-10
More Dirichlet eigenfunctions: D-19
More Dirichlet eigenfunctions: D-42
Refinement

- On level $m + 1$ we expect to see all $8^m$ eigenfunctions from level $m$ but refined.
- The eigenvalue is renormalized by $\rho = 10.011^{-1}$.

**Figure:** \( \phi_5^{(4)} \) and \( \phi_5^{(5)} \) with respective eigenvalues \( \lambda_5^{(4)} = 0.00328 \) and \( \lambda_5^{(4)} = 0.000328 \).
Miniaturization

Any level $m$ eigenfunction and eigenvalue miniaturizes on the level $m + 1$ carpet. It will consist of 8 copies of $\phi^{(4)}$ or $-\phi^{(4)}$.

Figure: $\phi^{(4)}_4$ and $\phi^{(5)}_{20}$ with respective eigenvalues $\lambda^{(4)}_4 = \lambda^{(5)}_{20} = 0.00177$. 
Describing the eigenvalue data

*Eigenvalue counting function:* $N(t) = \#\{\lambda : \lambda \leq t\}$

- $N(t)$ is the number of eigenvalues less than or equal to $t$. Describes the spectrum of eigenvalues.
- We expect the $N(t)$ to asymptotically grow like $t^\alpha$ as $t \to \infty$ where $\alpha = \log 8 / \log 10.011 \approx 0.9026$. 
$$N(t) = \# \{ \lambda : \lambda \leq t \}$$
Weyl Ratio: \( W(t) = \frac{N(t)}{t^\alpha} \quad \alpha \approx 0.9026 \)
By the min-max property, we can say that $\lambda_j^{(N)} \leq \lambda_j^{(D)}$ for each $j$. Therefore, $N^{(D)}(t) \leq N^{(N)}(t)$.

What is the growth rate of $N^{(N)}(t) - N^{(D)}(t)$. We suspect there is some power $\beta$ such that $N^{(N)}(t) - N^{(D)}(t) \sim t^\beta$.

$\beta = \frac{\log 3}{\log \rho} \approx 0.4769$
$N^{(N)}(t) - N^{(D)}(t)$
A stronger periodicity is apparent here.
We can eliminate the boundary of SC by gluing its boundary in specific orientations. We examined three types of SC fractafolds:
Some Eigenfunctions for the Fractafoils
Neumann - Torus counting function

$\beta \sim 0.4840$
The difference between the Neumann and Torus eigenvalue counting functions grow at a rate of

\[ t^\beta \quad \text{with} \quad \beta \approx 0.4769 \]

This is consistent for the Neumann-Dirichlet counting function. This estimate for \( \beta \) is \( \frac{\log 3}{\log \rho} \approx 0.4769 \).

(Similar results for Neumann minus Klein bottle, Projective space)
Neumann - Torus Weyl Ratio

![Graph showing Neumann-Torus Weyl Ratio](image)

$\beta \approx 0.4769$
Torus - Klein counting function
Torus - Projective counting function
Also note the rapid oscillation between positive and negative values.

For the difference of the torus and Klein bottle eigenfunctions (as well as Torus-Projective space) we conjecture that the difference is uniformly bounded. (Not enough data to distinguish between boundedness and log $t$ growth)
Ideas for defining a normal derivative

Analog of the Gauss-Green formula for functions $u$ satisfying Neumann boundary conditions:

$$E_m(u, v) = \frac{1}{\rho^m} \sum_x v(x) \sum_{y \sim x} (u(x) - u(y))$$

$$= - \sum_x 8^{-m} v(x) \sum_{y \sim x} \left(\frac{8}{\rho}\right)^m (u(y) - u(x))$$

$$= - \sum_x 8^{-m} v(x) \Delta_m u(x).$$

In the limit as $m \to \infty$ we obtain

$$E(u, v) = - \int_{SC} v \Delta u \, d\mu.$$

If $u$ does not satisfy Neumann boundary conditions then we do not have the correct expression for $\Delta_m u(x)$ when $x$ is a boundary cell, because we need to take into account the values when $y = x^*$, the virtual neighbor(s).
Ideas for defining a normal derivative

Thus we have

$$E_m(u, v) = - \sum_x 8^{-m} v(x) \Delta_m u(x) + \sum_{x \in \partial SC_m} v(x) \rho^{-m} (u(x) - u(x^*)) .$$

In the limit we hope to obtain

$$E(u, v) = - \int_{SC} v \Delta u \, d\mu + \int_{\partial SC} v \partial_n u \, d\nu$$

for some appropriate measure on the boundary, and some appropriate definition of the normal derivative $\partial_n u$. Since the boundary of SC is the same as the boundary of the square, one might speculate that $d\nu$ could just be Lebesgue measure, but our data does not support this supposition, and indeed there is no reason to believe that the measure should be so place independent, since the geometry of SC near boundary points varies considerably from point to point.

We are not prepared to put forward a conjectural definition of the normal derivative.
More data is available (and more to come) on www.math.cornell.edu/~reu/sierpinski-carpet including:

- List of eigenvalues and pictures of eigenfunctions for both Dirichlet and Neumann boundary value problems.
- Eigenvalue counting function data & Weyl ratios.
- Eigenvalue data on fractafolds and covering spaces of SC.
- Trace of the Heat Kernel data.
- Dirichlet and Poisson kernel data.
- All MATLAB scripts used.