Big numbers, graph coloring, and Hercules' battle with the hydra

Cornell University

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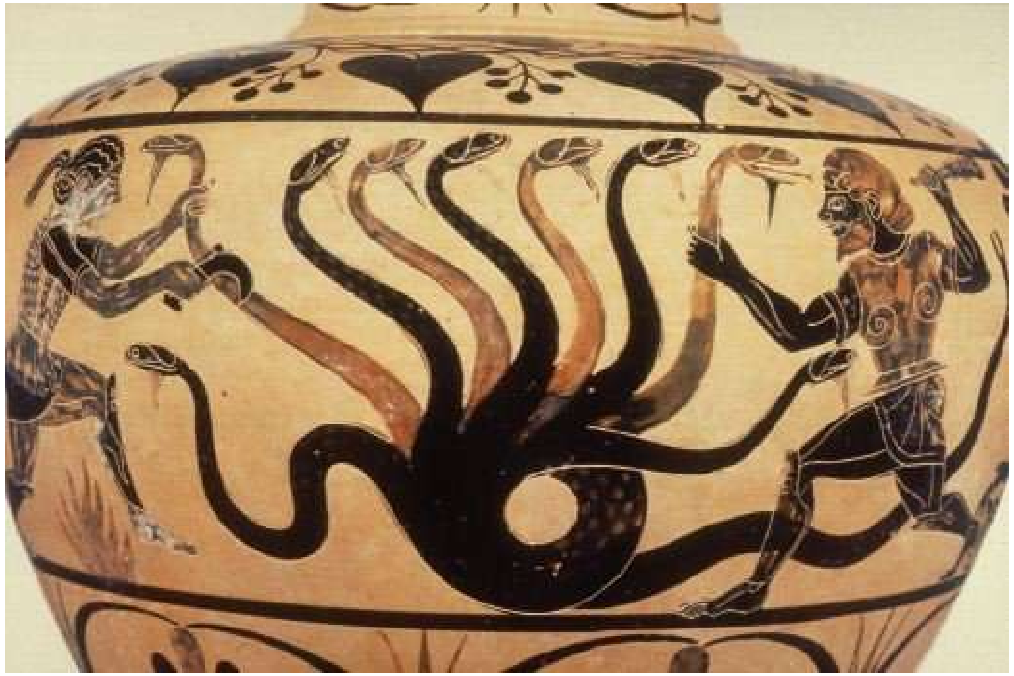
March 12, 2011



K–12 Education and Outreach

Challenge

You have two minutes. Using standard math notation, English words, or both, name a single whole number—not an infinity—on a blank index card. Be precise enough for any reasonable modern mathematician to determine exactly what number you've named, by consulting only your card and, if necessary, the published literature.

THE LARGEST OF ALL YOUR
NUMBERS, PLUS ONE.

Number of zeros	U.S. & scientific community	Other countries
3	thousand	thousand
6	million	million
9	billion	1000 million (1 milliard)
12	trillion	billion
15	quadrillion	1000 billion
18	quintillion	trillion
21	sextillion	1000 trillion
24	septillion	quadrillion
27	octillion	1000 quadrillion
30	nonillion	quintillion
33	decillion	1000 quintillion
36	undecillion	sextillion
39	duodecillion	1000 sextillion
42	tredecillion	septillion
45	quattuordecillion	1000 septillion
48	quindecillion	octillion
51	sexdecillion	1000 octillion
54	septendecillion	nonillion
57	octodecillion	1000 nonillion
60	novemdecillion	decillion
63	vigintillion	1000 decillion
66 - 120		undecillion - vigintillion
303	centillion	
600		centillion

A googol = 10^{100}

A googolplex = 10^{googol}

"yammkosioi" (Greek)

= "sand – hundred"

Pindar: 'sand escapes counting"

Pindar: ca. 522 – 443 BC

A vigintillion

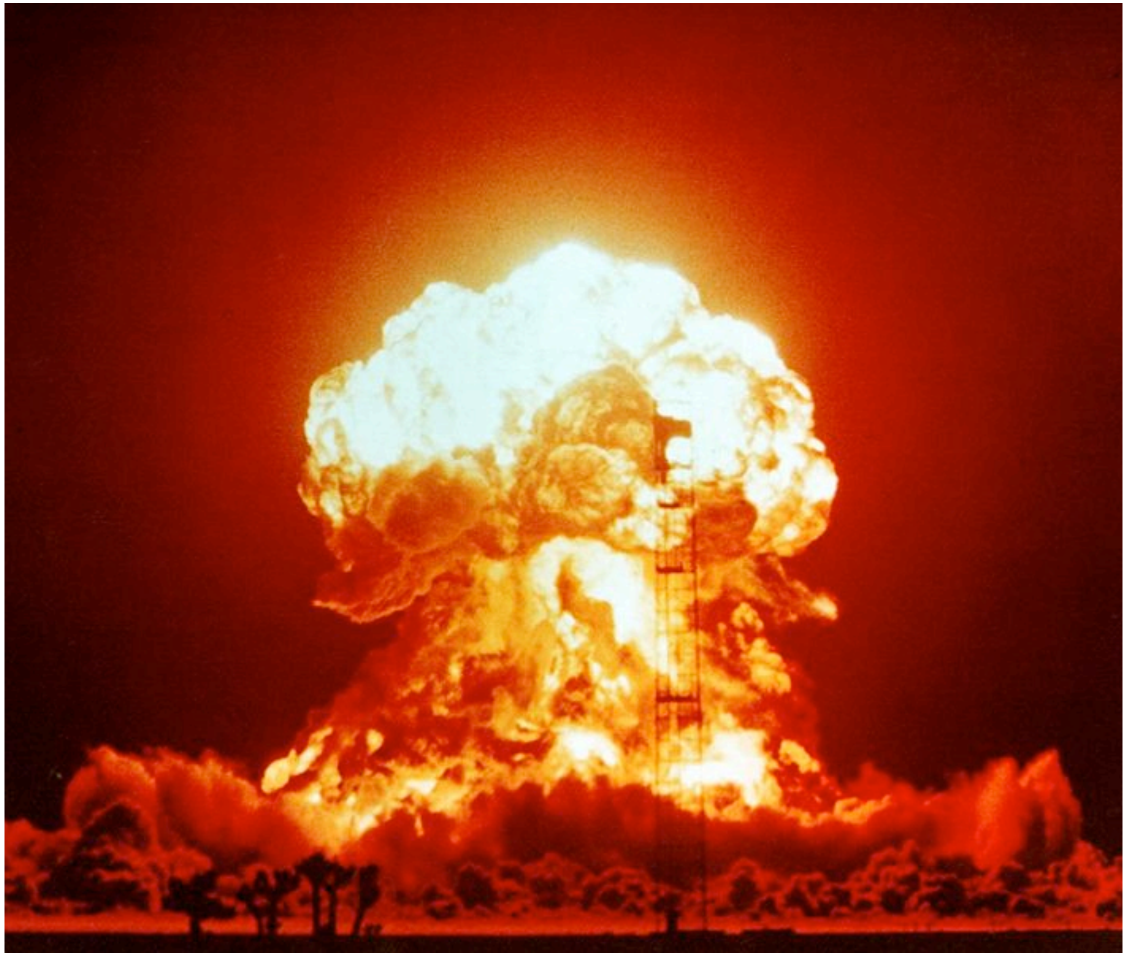
10^{63}

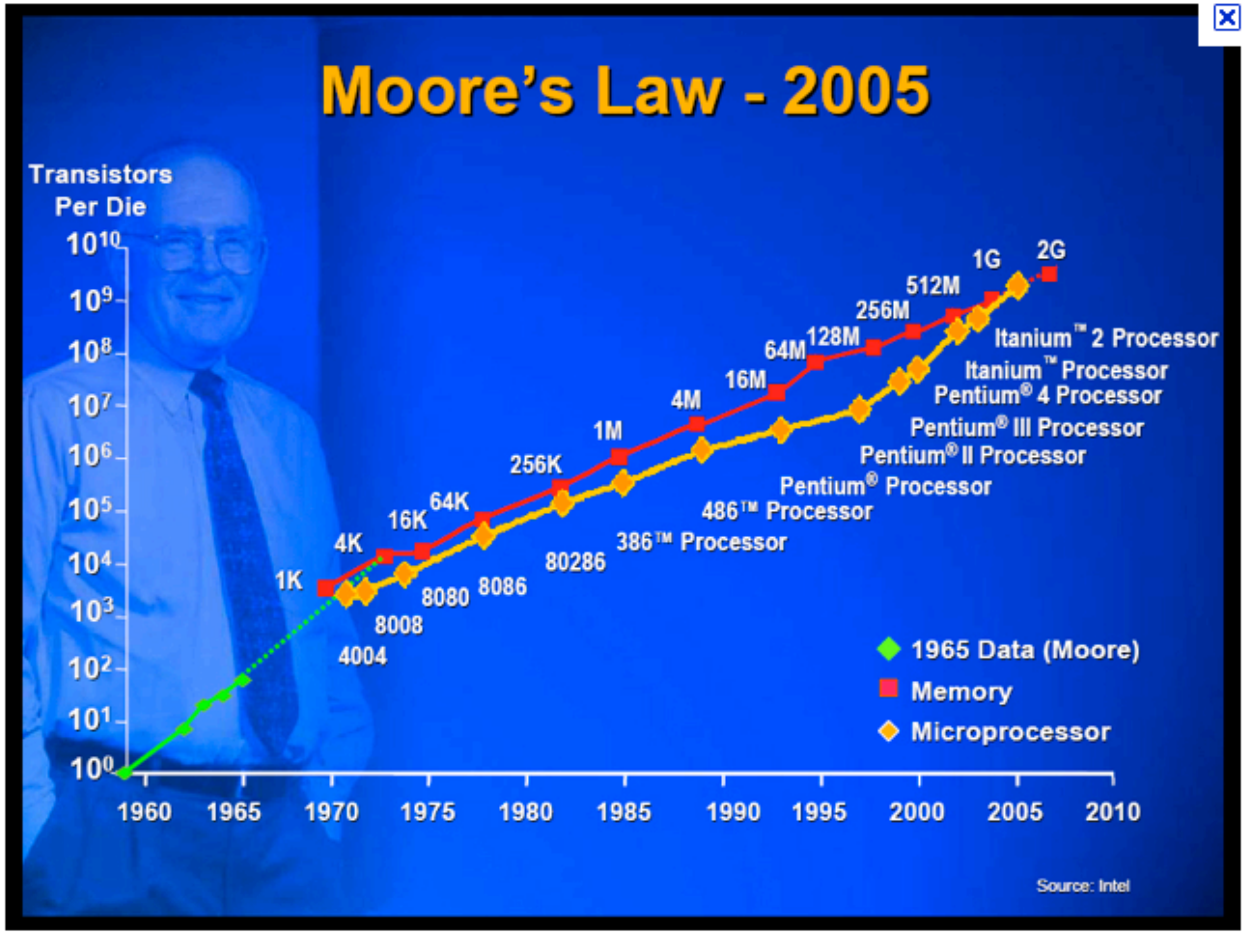
"The Sand-Reckoner"

Archimedes
c. 287 BC – c. 212 BC

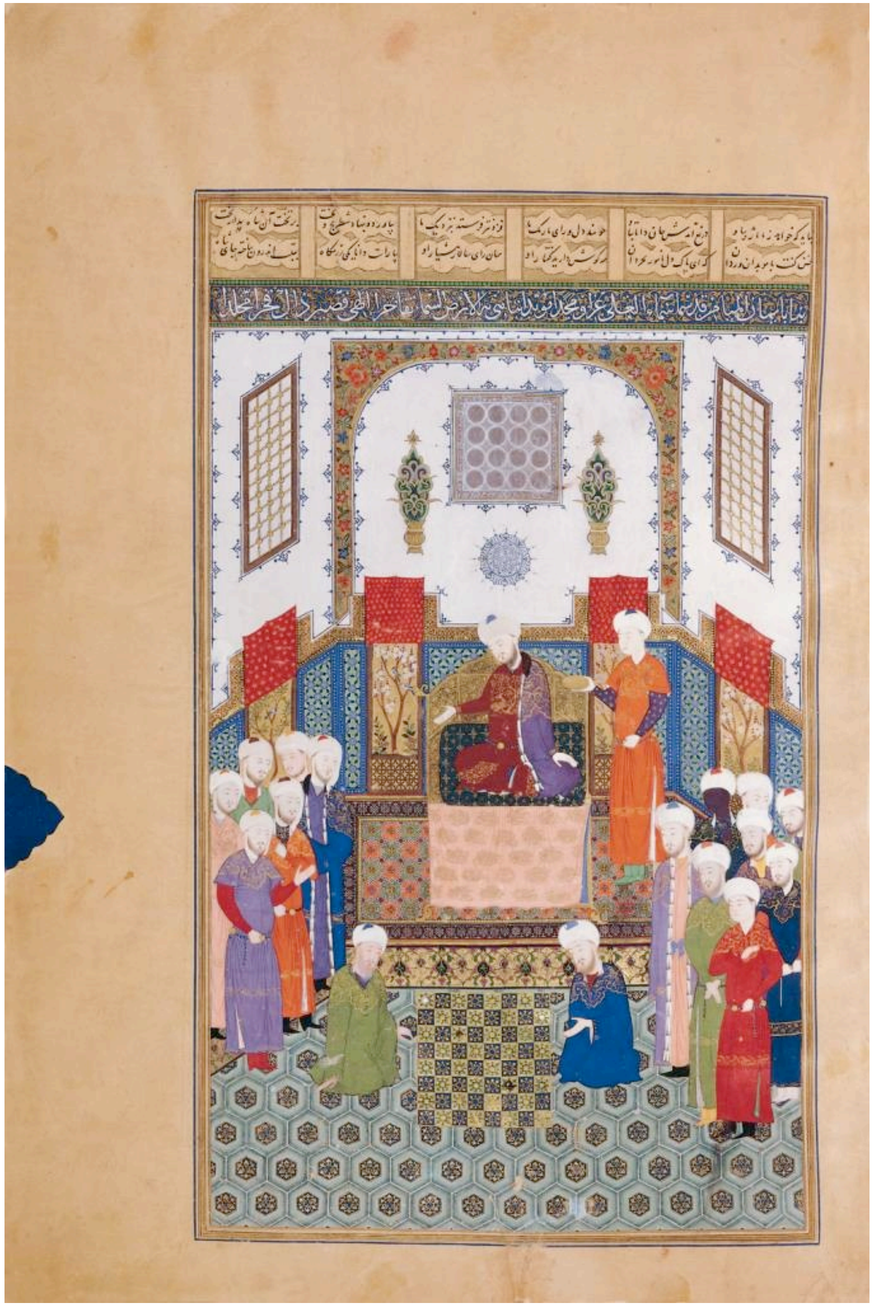
920,000,000,000,000,000,000,000,000	9.2×10^{26}	Approximate diameter (in metres) of the visible universe (92 billion light years).
1,000,000,000,000,000,000,000,000,000	1×10^{27}	Temperature (in ° Kelvin) of the universe 10^{-35} seconds after the Big Bang, at the start of the inflationary epoch.
2,000,000,000,000,000,000,000,000,000,000	2×10^{30}	Mass (in kilograms) of the Sun (1 solar mass).
40,000,000,000,000,000,000,000,000,000,000	4×10^{31}	Mass (in kilograms) of Betelgeuse, a red supergiant star (about 20 solar masses).
141,700,000,000,000,000,000,000,000,000,000	1.417×10^{32}	Planck Temperature, the temperature (in ° Kelvin) of the universe at 1 Planck Time after the Big Bang.
10,000,000,000,000,000,000,000,000,000,000,000,000,000	1×10^{40}	Approximate ratio of the strength of the electromagentic to the gravitational force between sub-atomic particles.
36,000,000,000,000,000,000,000,000,000,000,000,000,000	3.6×10^{40}	Mass (in kilograms) of OJ287, the largest measured supermassive black hole.
687,000,000,000,000,000,000,000,000,000,000,000,000,000	6.87×10^{41}	Gravitational binding energy (in Joules) of the Sun.
120,000,000,000,000,000,000,000,000,000,000,000,000,000,000	1.2×10^{44}	Estimated energy (in Joules) released in a supernova explosion.
30,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000	3×10^{52}	Estimated mass (in kilograms) of the observable universe.
4,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000	4×10^{69}	Estimated total mass-energy (in Joules) of the observable universe.
100,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000	1×10^{80}	Estimate the total number of fundamental particles in the observable universe (other estimates go up to 10^{85}).
5,100,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000	5.1×10^{96}	Planck density, the density (in kg/metre^3) of the universe at one unit of Planck time after the Big Bang.

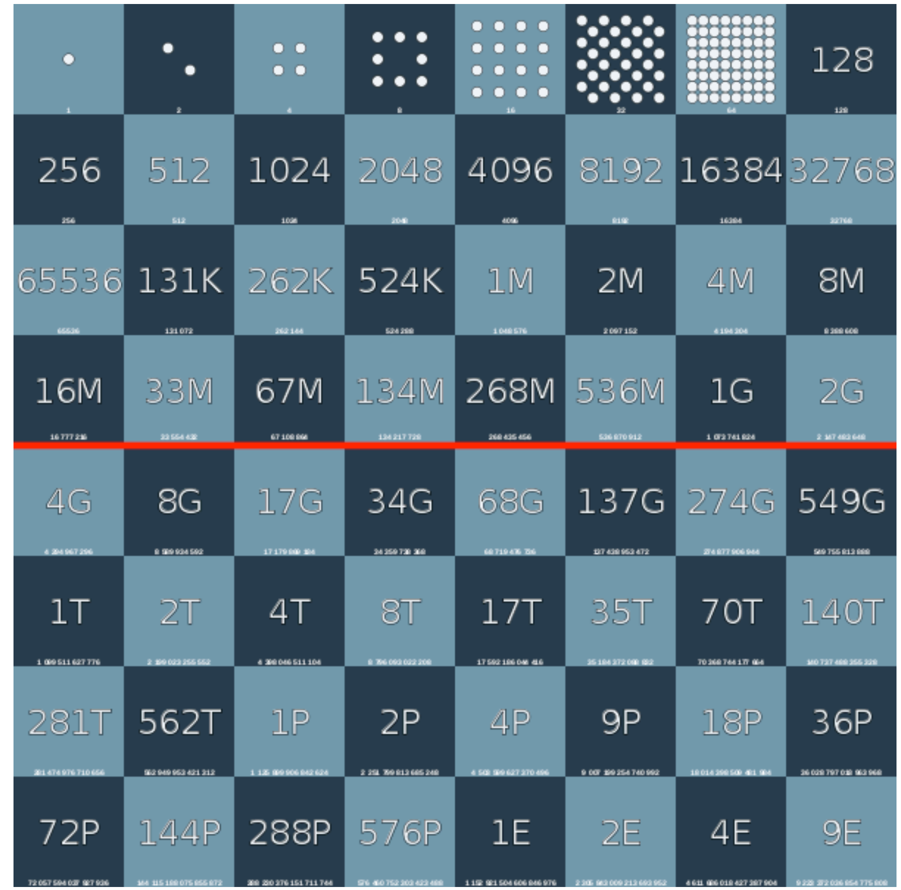
Exponential growth





Date	Drive info	Size	Cost	$/GB
January 1980	Morrow Systems	26MB	$5,000.00	$193,000.00
March 1989	Western Digital	40MB	$1,199.00	$36,000.00
February 12, 1999	Quantum	8GB	$299.99	$43.10
July 24, 2009	HITACHI 0A38016 7200 RPM SATA 3.0Gb/s	1TB	$74.99	$0.07

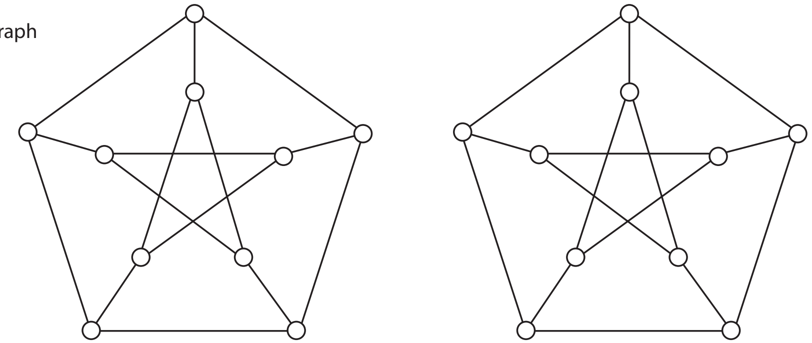




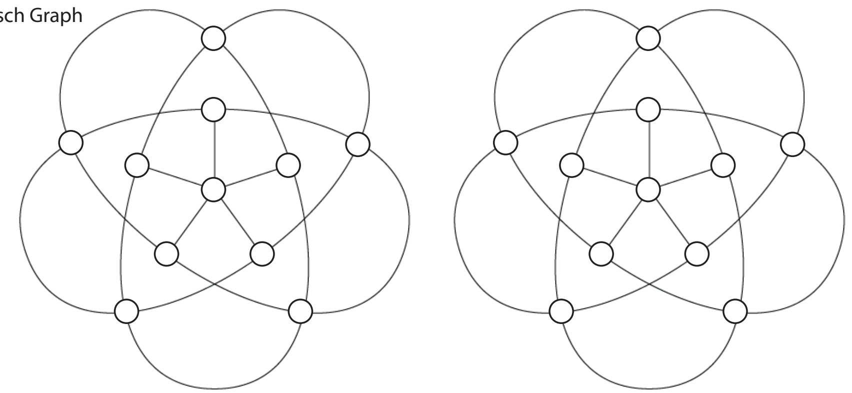
Bozorgmehr, King Anushirvan of Persia's grand vizier, challenges the Indian envoy to a game of chess

Graph coloring and chromatic number

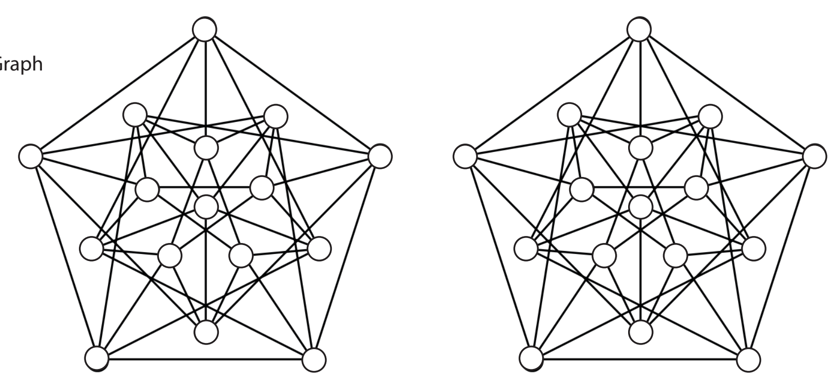
Petersen Graph



Groetzsch Graph

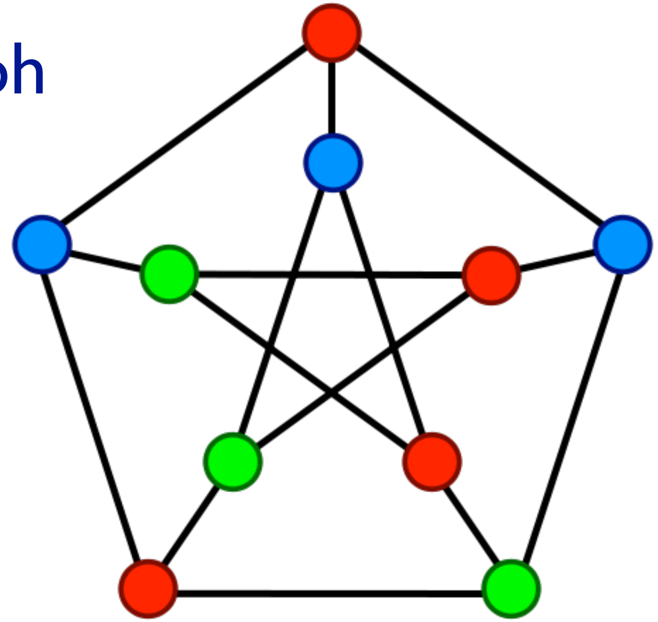


Clebsch Graph



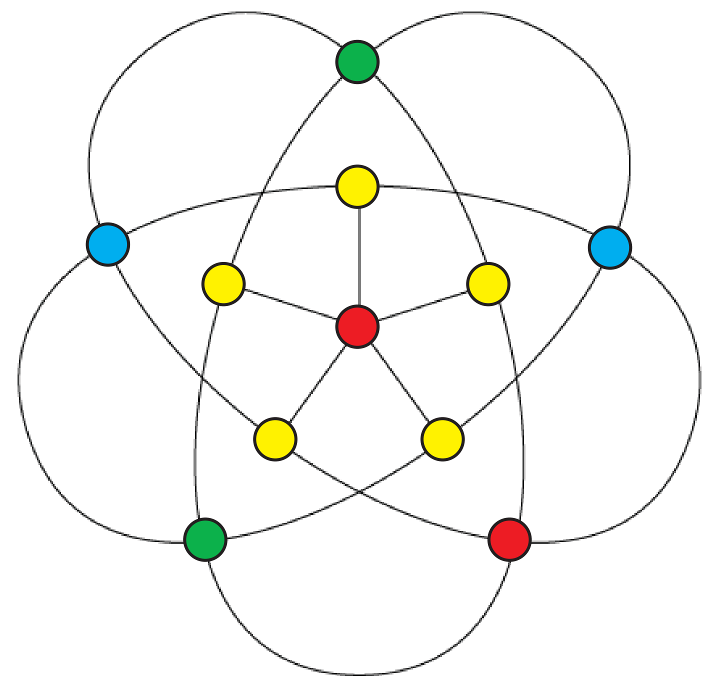
Petersen Graph

chromatic number = 3



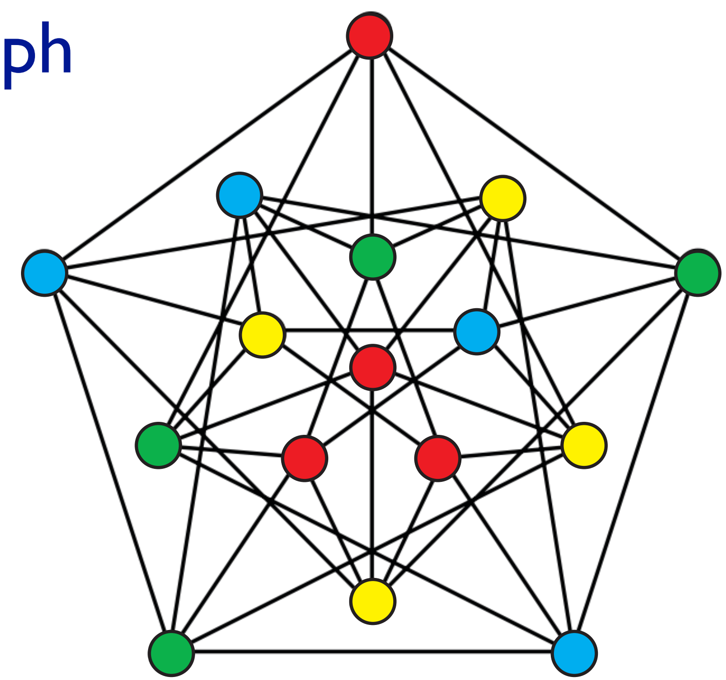
Groetzsch Graph

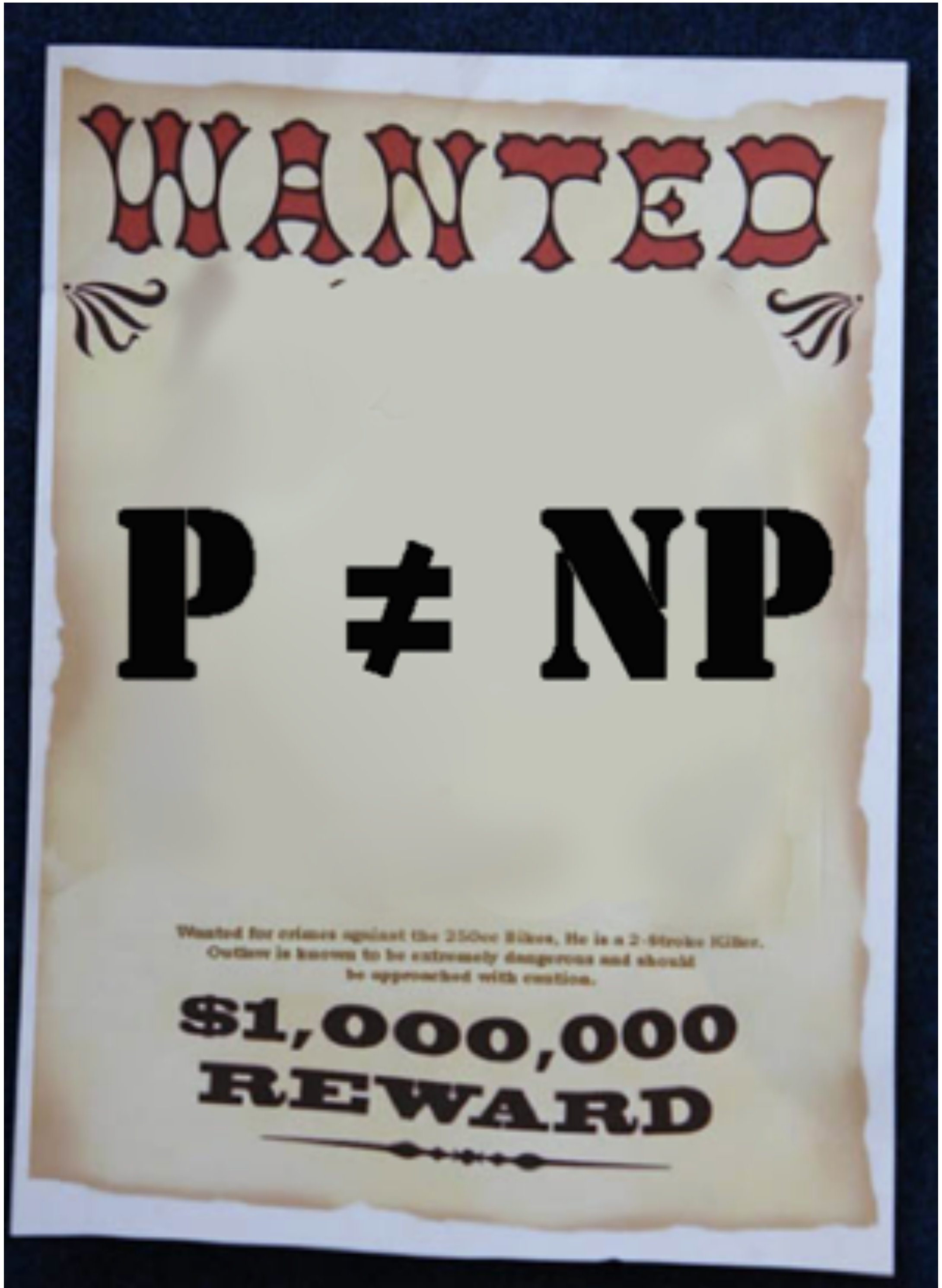
chromatic number = 4



Clebsch Graph

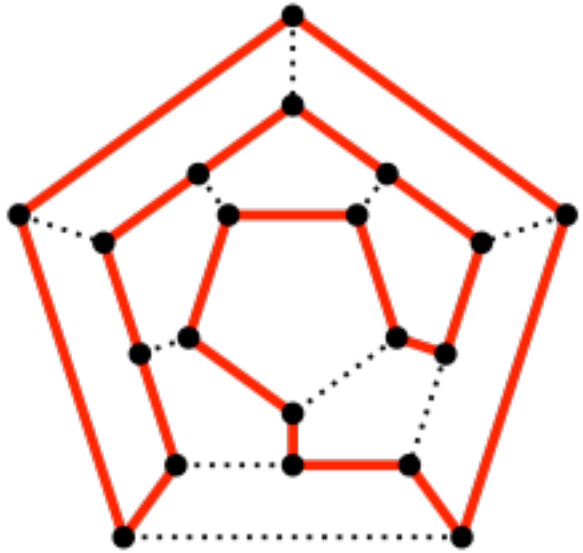
chromatic number = 4



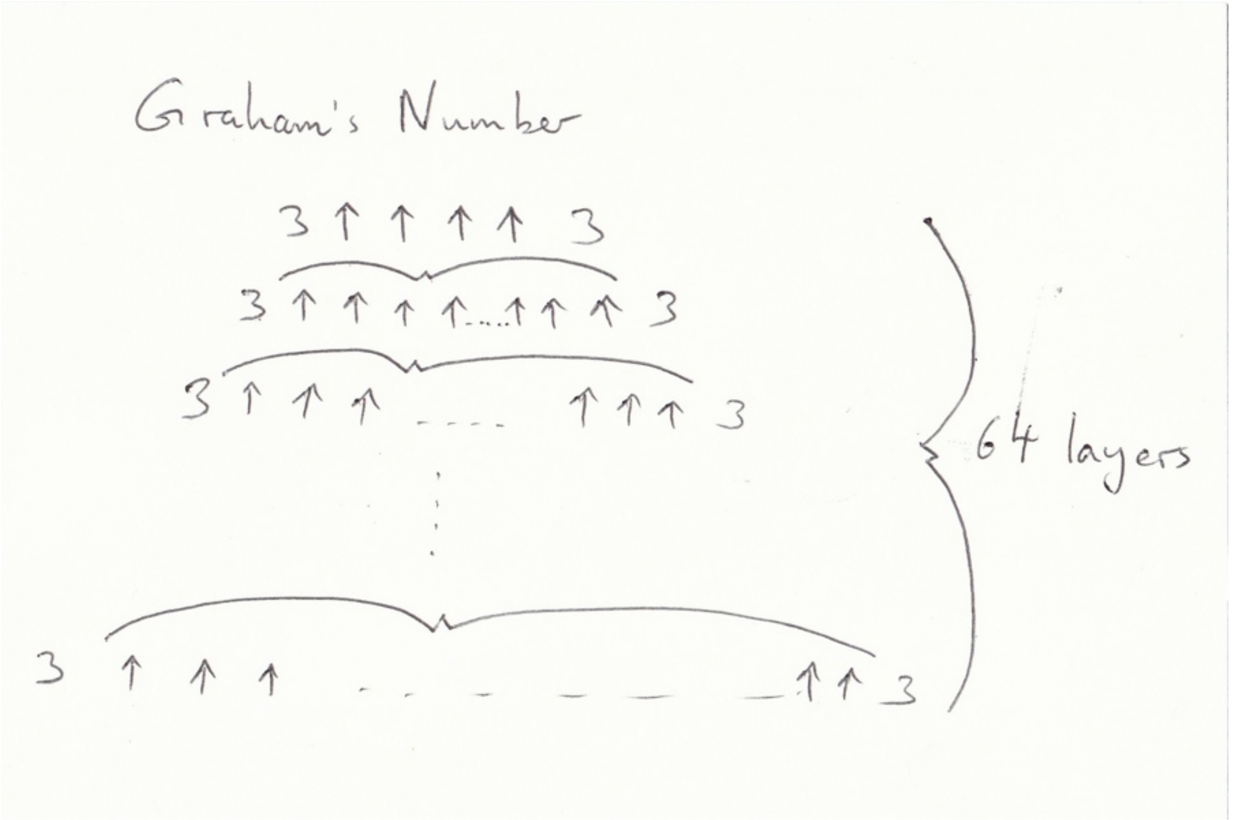


Is there a polynomial time algorithm that will tell you whether a graph is 3-colorable?

- Subset sum
- Hamiltonian cycles
- Travelling salesman problem



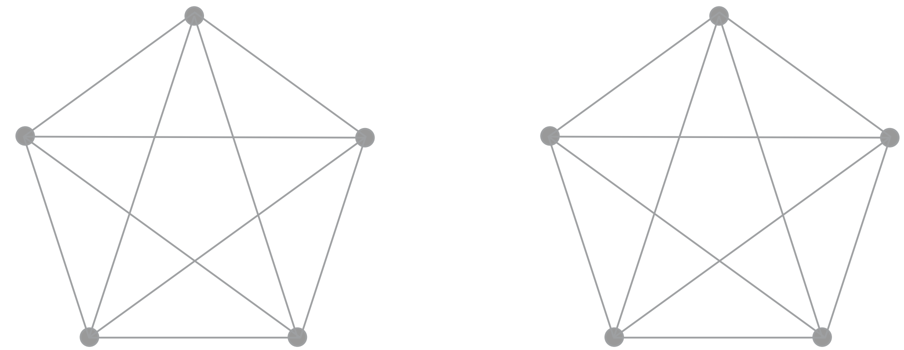
Graham's number and Ramsey Theory



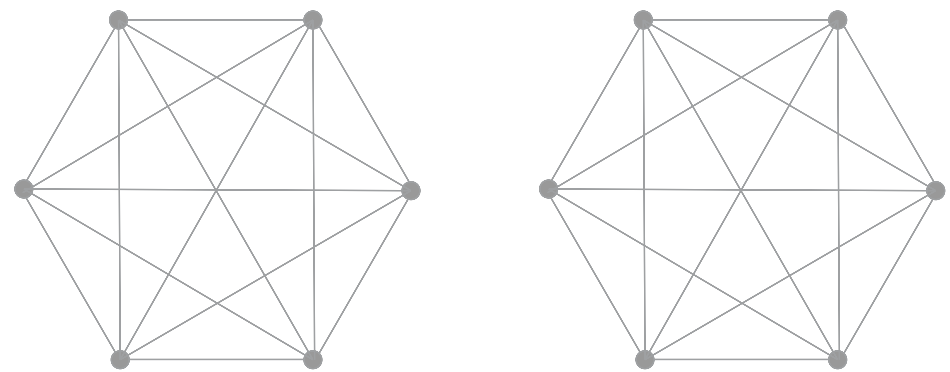
Ramsey Theory game

“Sim”

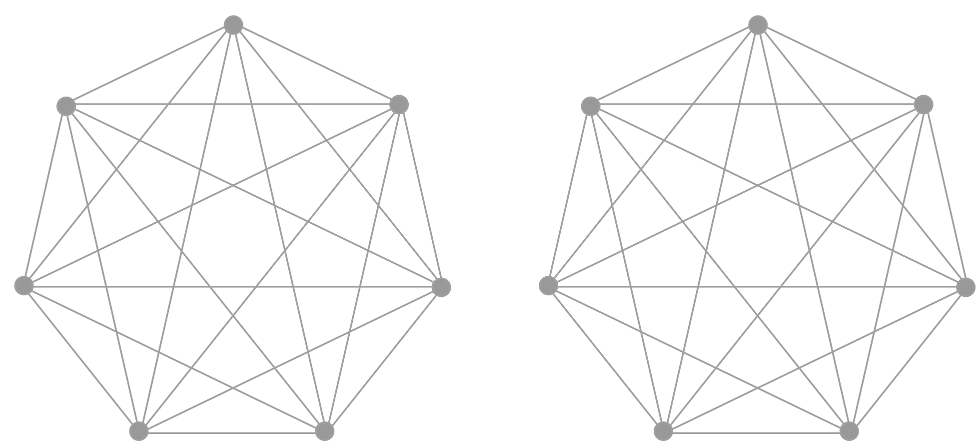
Guatsavo Simmons 1969



K_5

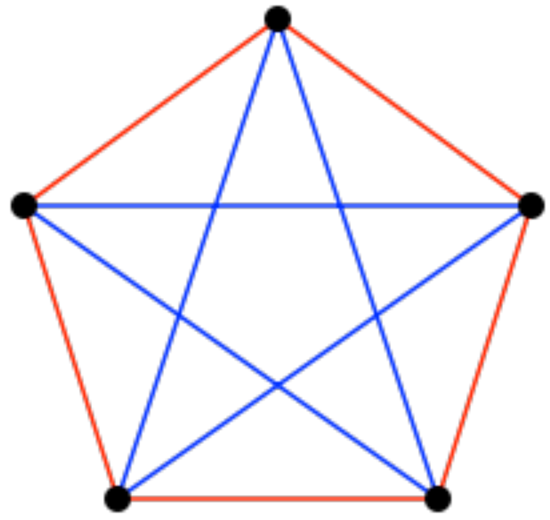


K_6



K_7

On K_5, there can be draws.



K_6 – *At a gathering of any six people, some three of them are either mutual acquaintances or complete strangers.*

There are no draws.

On K_6, with perfect strategy, the second player always wins.

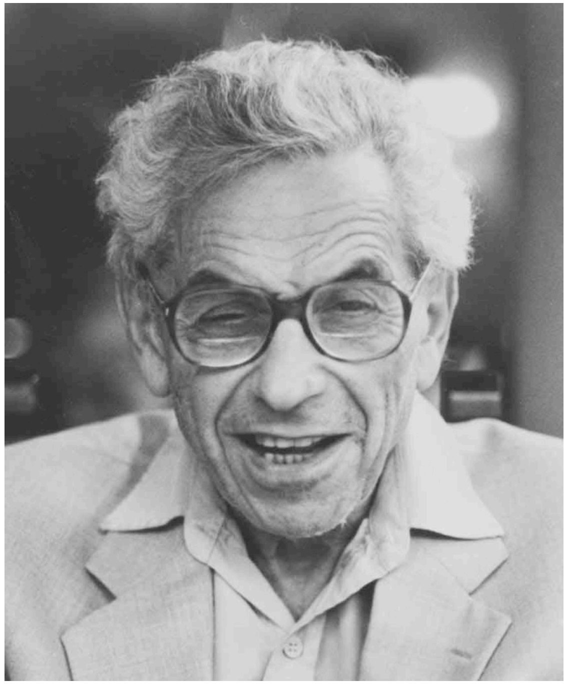
$R(m)$ is the minimal n such that however one 2–colors K_n, it will always contain a monochrome K_m subgraph.

$$R(1) = 1 \qquad R(4) = 18$$

$$R(2) = 2 \qquad 43 \leq R(5) \leq 49$$

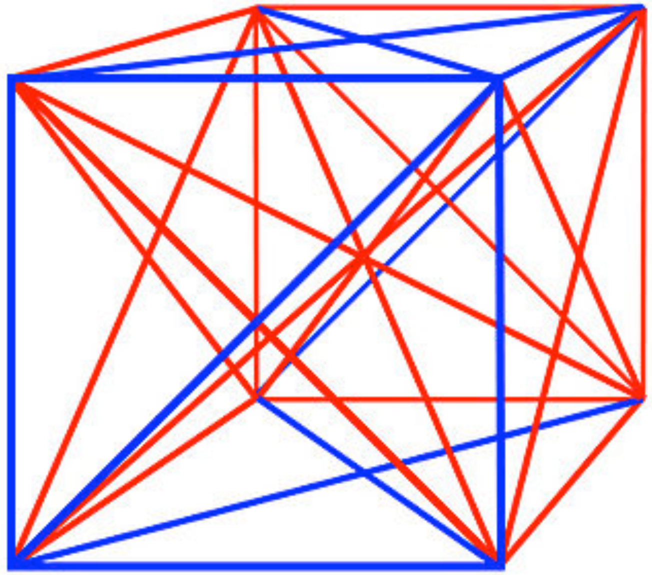
$$R(3) = 6 \qquad 102 \leq R(6) \leq 165$$

"Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6)$. In that case, he believes, we should attempt to destroy the aliens."

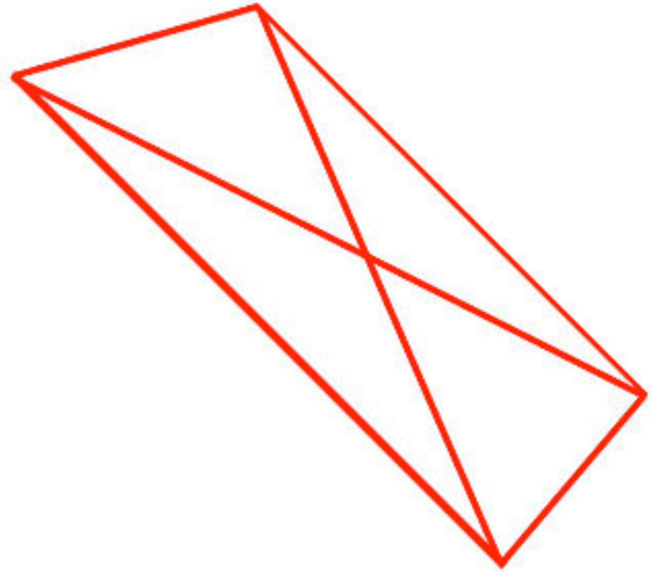


Paul Erdős, 1913 – 1996

Consider an n–dimensional hypercube, and connect each pair of vertices to obtain K_{2^n} .



What is the smallest value of n such that every 2–coloring contains at least one monochrome planar K_4 subgraph?



Knuth's arrow notation

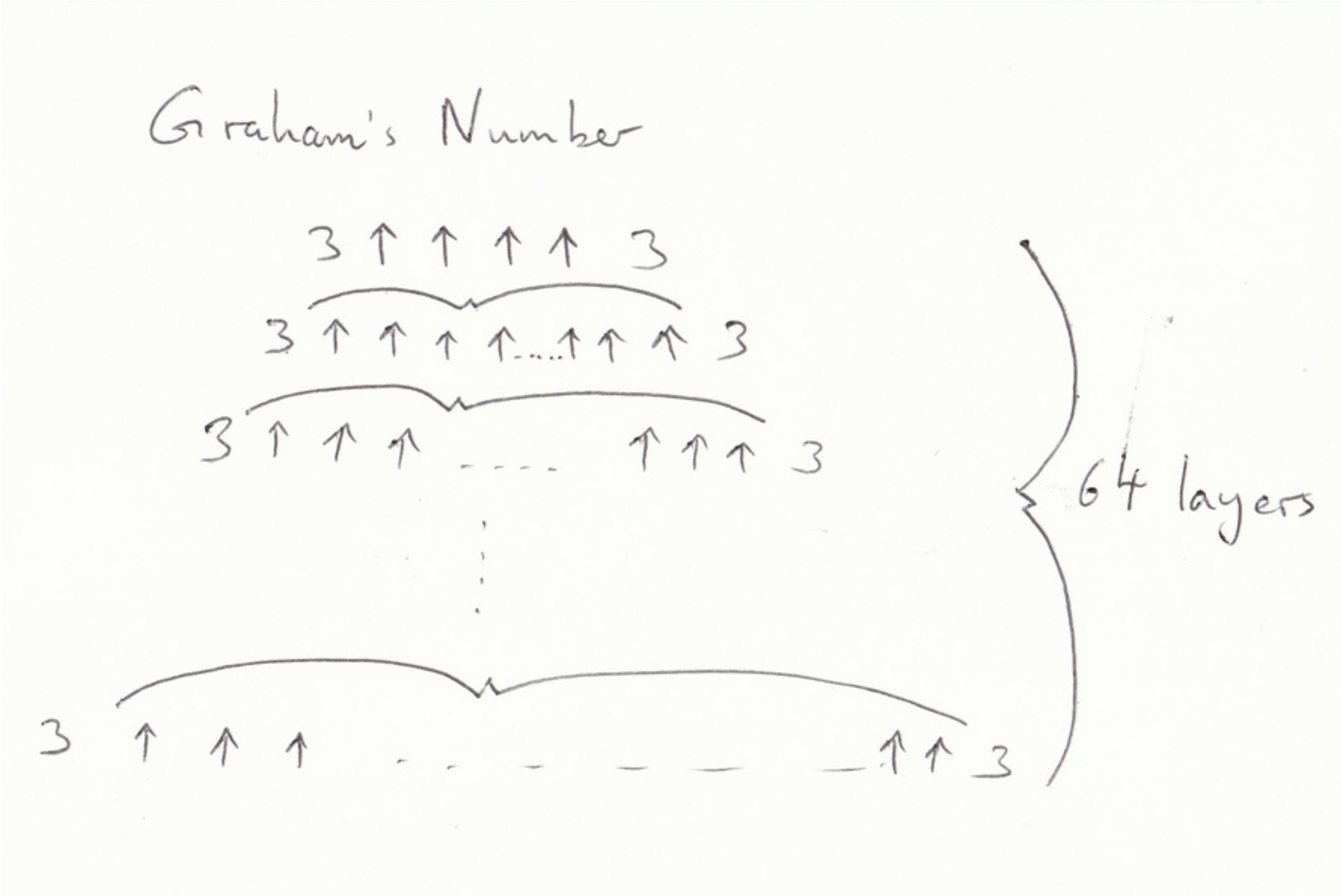
$$a \uparrow b = a^b = \underbrace{a \times a \times \cdots \times a}_{b \text{ copies of } a}$$

$$a \underbrace{\uparrow\uparrow \ldots \uparrow}_{n} b = \underbrace{a \underbrace{\uparrow \ldots \uparrow}_{n-1} a \underbrace{\uparrow \ldots \uparrow}_{n-1} a \ \ldots \ a \underbrace{\uparrow \ldots \uparrow}_{n-1} a}_{b \text{ copies of } a}$$ evaluated right to left.

Example

$$3 \uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow 3 \uparrow\uparrow 3 = 3 \uparrow\uparrow (3 \uparrow 3 \uparrow 3) = \underbrace{3 \uparrow 3 \uparrow \ldots \uparrow 3}_{3 \uparrow 3 \uparrow 3 \text{ copies of } 3}$$

$$= \underbrace{3 \uparrow 3 \uparrow \ldots \uparrow 3}_{7{,}625{,}597{,}484{,}987 \text{ copies of } 3}$$



is an upper bound for the solution to the hypercube problem.

General belief is that the answer to Graham's problem is 6.

"The infinite we shall do right away. The finite may take a little longer."

Stanislaw Ulam


HERCVLES VNA CVM IOLAO HYDRAM OCCIDIT.
15 HB 45

A *hydra* is a word on letters a and b.

- Hercules strikes off the first letter.
- The hydra regenerates by:

$$a \mapsto ab$$
$$b \mapsto b$$

Repeat.

- *Who wins?*
- *How long does it take?*
- *Generalisations? Questions?*

Example

$b\,a\,b\,b\,a\,b$
$a\,b\,b\,b\,a\,b\,b$
$b\,b\,b\,a\,b\,b\,b$
$b\,b\,a\,b\,b\,b\,b$
$b\,a\,b\,b\,b\,b\,b$
$a\,b\,b\,b\,b\,b\,b$
$b\,b\,b\,b\,b\,b$
$b\,b\,b\,b\,b$
$b\,b\,b\,b$
$b\,b\,b$
$b\,b$
b

Hercules is victorious in 12 strikes.

a

1 strike

$a\,a$
$a\,b$
b

3 strikes

$a\,a\,a$
$a\,b\,a\,b$
$b\,a\,b\,b$
$a\,b\,b\,b$
$b\,b\,b$
$b\,b$
b

7 strikes

$a\,a\,a\,a$
$a\,b\,a\,b\,a\,b$
$b\,a\,b\,b\,a\,b\,b$
$a\,b\,b\,b\,a\,b\,b\,b$
$b\,b\,b\,a\,b\,b\,b\,b$
$b\,b\,a\,b\,b\,b\,b\,b$
$b\,a\,b\,b\,b\,b\,b\,b$
$a\,b\,b\,b\,b\,b\,b\,b$
$b\,b\,b\,b\,b\,b\,b$
$b\,b\,b\,b\,b\,b$
$b\,b\,b\,b\,b$
$b\,b\,b\,b$
$b\,b\,b$
$b\,b$
b

15 strikes

$\underbrace{a\,a\cdots a}_{n}$

$\vdots$

$2^n - 1$ strikes

Hercules wins against all hydra.

Hydra using three letters a, b and c.

Regeneration: $a \mapsto ab$, $b \mapsto bc$, $c \mapsto c$.

---

a

1 strike

aa
ab
bc
c

4 strikes

aaa
$abab$
$bcabbc$
$cabbcbcc$
$abbcbccbccc$
$bcbccbcccbcccc$
$cbcccbccccbccccc$
$bccccbcccccbcccccc$
$ccccbccccccbccccccc$
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$cccccc$
$ccccc$
$cccc$
ccc
cc
c

46 strikes

$aaaa$
$ababab$
$bcabbcabbc$
$cabbcbccabbcbcc$
$abbcbccbcccabbcbccbccc$
⋮

211106232532990 strikes

$\mathcal{H}(w) :=$ # strikes it takes to kill w.

$$\mathcal{H}(a^{n+1}) = 3 \cdot 2^{\mathcal{H}(a^n)} - 2$$

Hercules still wins against all hydra... eventually!

Hydra using four letters a, b, c and d.

Regeneration: $a \mapsto ab$, $b \mapsto bc$, $c \mapsto cd$, $d \mapsto d$.

---

a

1 strike

aa
ab
bc
cd
d

5 strikes

aaa
$abab$
$bcabbc$
$bccdabbccd$
$cdcddabbcbccdcdd$
$\vdots$

$3 \cdot 2^{3 \cdot 2^{3 \cdot 2^{95}-1}-1} - 1$ strikes

Hercules still always wins!

$$a_1, a_2, \ldots$$

$$a_i \mapsto a_i a_{i-1}, \forall i > 1$$

$$a_1 \mapsto a_1$$

$$\mathcal{H}_k(n) := \mathcal{H}({a_k}^n) < \infty$$

$k \backslash n$	1	2	3	4	$\cdots$	n	$\cdots$
1	1	2	3	4	$\cdots$	n	$\cdots$
2	1	3	7	15	$\cdots$	$2^n - 1$	$\cdots$
3	1	4	46	211106232532990	$\cdots$		
4	1	5	$3 \cdot 2^{3 \cdot 2^{3 \cdot 2^{95} - 1} - 1} - 1$	$\vdots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$				

Ackermann's function. For integers $k, n > 0$,

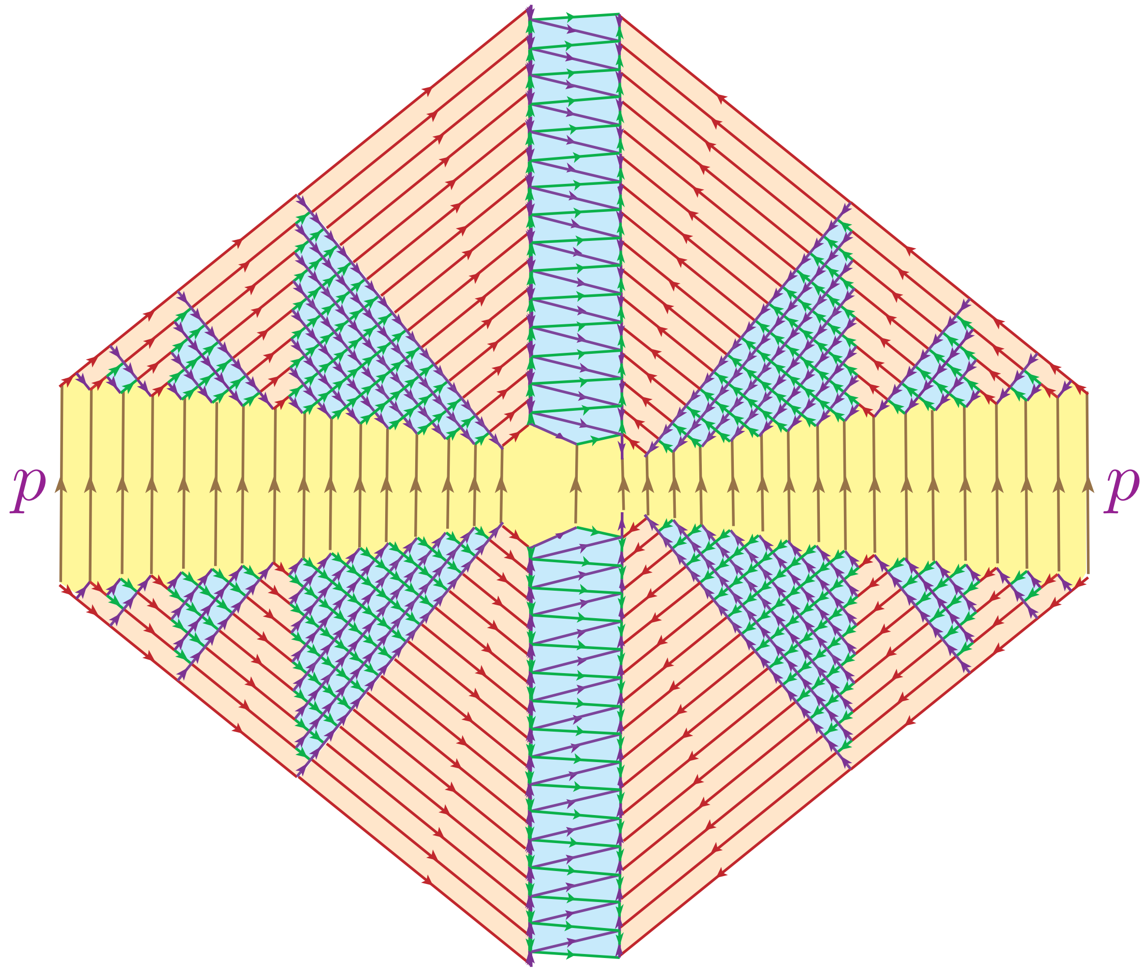
$$A_1(n) := 2n \quad \text{and} \quad A_{k+1}(n) := A_k^{(n)}(1).$$

$k \backslash n$	1	2	3	4	5	$\cdots$	n	$\cdots$
1	2	4	6	8	10	$\cdots$	$2n$	$\cdots$
2	2	4	8	16	32	$\cdots$	2^n	$\cdots$
3	2	4	16	65536	2^{65536}	$\cdots$	$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}n$	$\cdots$
4	2	4	65536	$A_3(65536)$	$\cdots$	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				

$$\forall k \geq 3, n \geq 2,\ \mathcal{H}_k(n) \geq A_k(n)$$
$$\forall k \geq 1, n \geq 0,\ \mathcal{H}_k(n) \leq A_k(n+k)$$

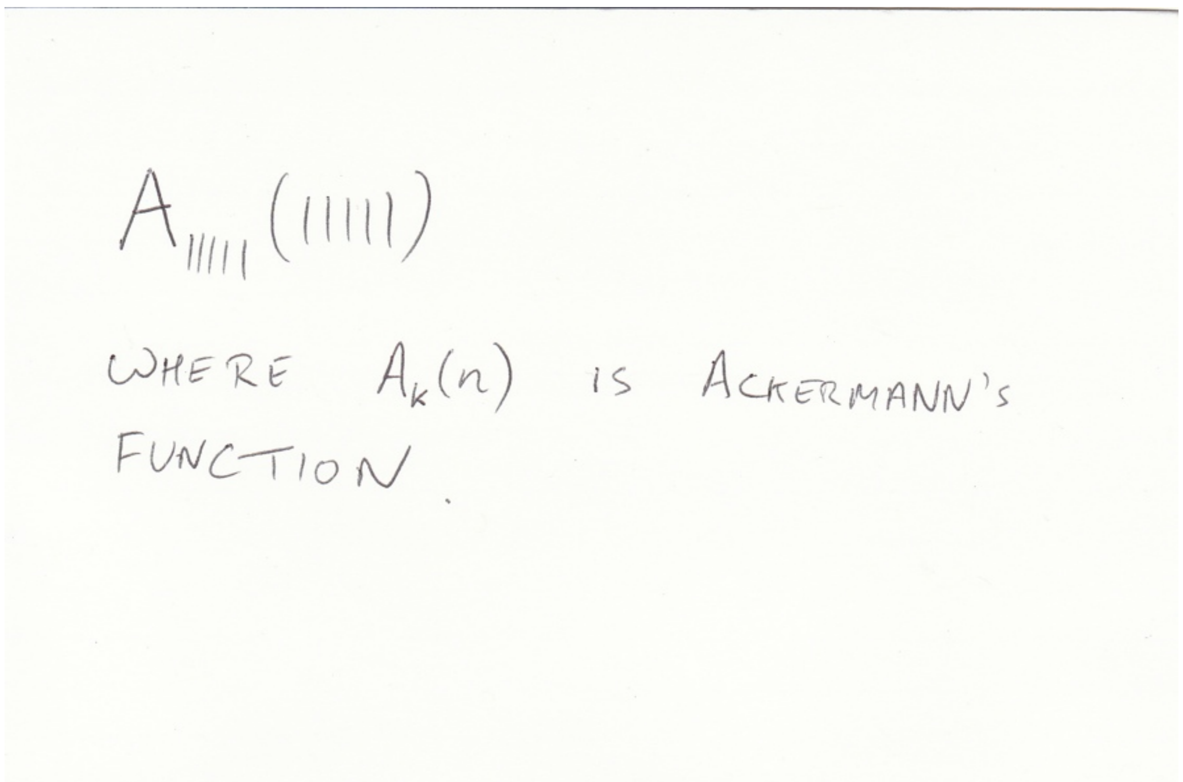
Theorem $\left\langle a_1, \ldots, a_k, t, p \;\middle|\; \begin{array}{l} t^{-1} a_1 t = a_1, \\ t^{-1} a_i t = a_i a_{i-1} \quad (i > 1) \\ [p, a_i t] = 1 \quad (i > 0) \end{array} \right\rangle$

$= \langle\, G_k, p \mid [p, H_k] \,\rangle$ has Dehn function $\simeq A_k$ when $k > 1$.



Going faster!

$n \mapsto A_n(n)$ is "recursive" but not "primitive recursive".



Wilhelm Ackermann, 1896 – 1962

Bump the base and subtract 1

$$
\begin{aligned}
266 &= 2^8 + 2^3 + 2 \\
6590 &= 3^8 + 3^3 + 2 \\
65601 &= 4^8 + 4^3 + 1 \\
390750 &= 5^8 + 5^3 \\
1679831 &= 6^8 + 5 \times 6^2 + 5 \times 6 + 5 \\
5765085 &= 7^8 + 5 \times 7^2 + 5 \times 7 + 4 \\
&\vdots \\
? &\quad ?
\end{aligned}
$$

Goodstein sequences

[Goodstein, 1944]

$$266 = 2^8 + 2^3 + 2 = 2^{2^{2+1}} + 2^{2+1} + 2$$

$$443426488243037769948249630619149892886 = 3^{3^{3+1}} + 3^{3+1} + 2$$

$$\begin{array}{l}32317006071311007300714876688669951960444102669715484032130345427524655138867 8\\ 908931972014115229134636887179609218980194941195591504909210950881523864482831\\ 206308773673009960917501977503896521067960576383840675682767922186426197561618\\ 380943384761704705816458520363050428875758915410658086075523991239303855219143\\ 333896683424206849747865645694948561760353263220580778056593310261927084603141\\ 502585928641771167259436037184618573575983511523016459044036976132332872312271\\ 256847108202097251571017269313234696785425806566979350459972683529986382155251\\ 663894373355436021354332296046453184786049521481935558536110595962316 81\end{array} = 4^{4^{4+1}} + 4^{4+1} + 1$$

$$5^{5^{5+1}} + 5^{5+1}$$

$$6^{6^{6+1}} + 5 \cdot 6^6 + 5 \cdot 6^5 + \cdots + 5 \cdot 6 + 5$$

$$\vdots$$

$$0$$

Reuben Louis Goodstein
1912 – 1985

Raymond Smullyan

Balls in a box.

A game of "bounded height, but unbounded width".

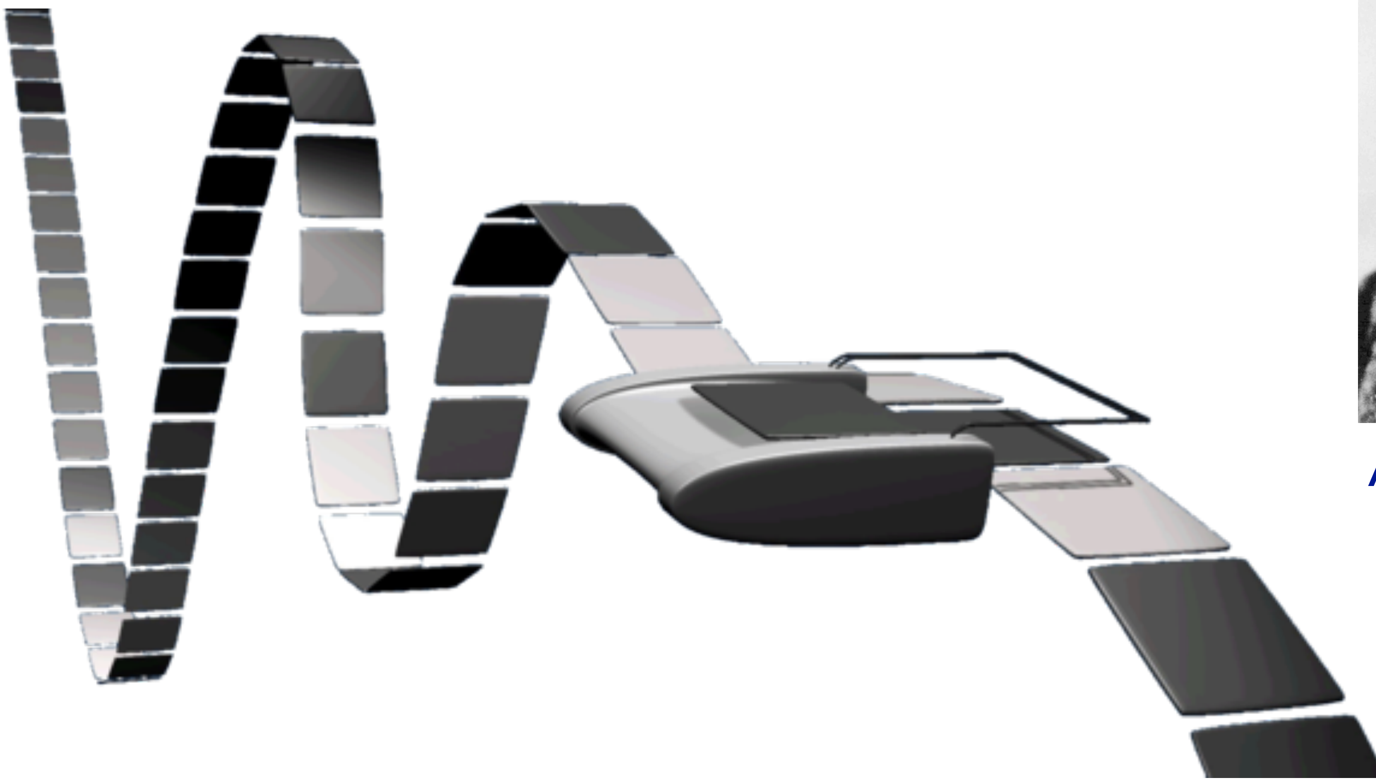
Gödel's First Incompleteness Theorem

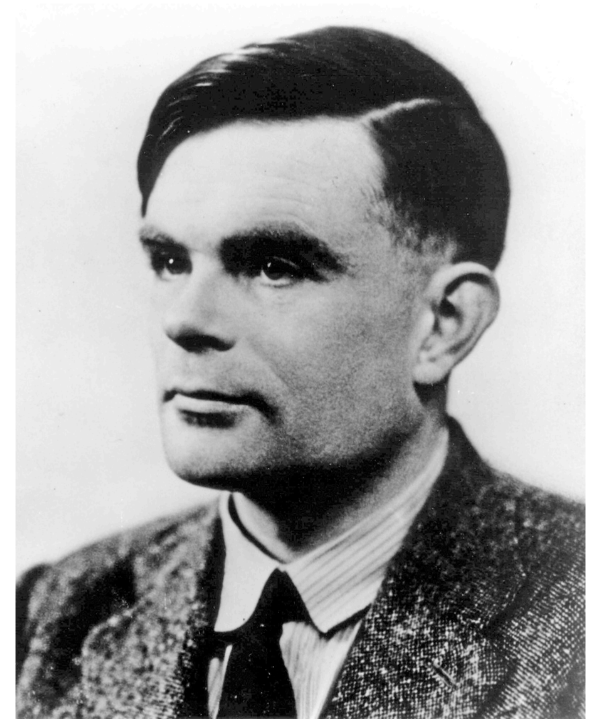
Kurt Gödel, 1906 – 1978

Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but not provable in the theory (*Kleene 1967, p. 250*).

Kirby & Paris (*1982*)
It is unprovable in Peano Arithmetic that all Goodstein sequences terminate at zero.

Turing Machines and Busy Beaver Functions





Alan Turing, 1912 – 1954

Radó's Busy Beaver function:

$BB(n)$ is the maximum halting time of all halting Turing machines of *size* at most n.

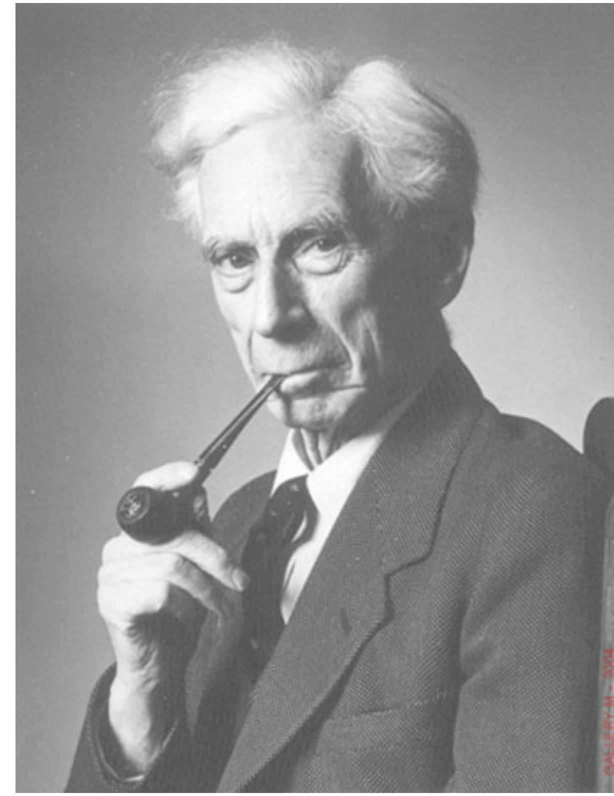
Tibor Radó, 1895 – 1965

BB(11111)

WHERE BB(n) IS THE BUSY BEAVER FUNCTION.

THE LARGEST WHOLE NUMBER
NAMEABLE WITH 1,000
CHARACTERS OF ENGLISH TEXT.

ONE PLUS THE LARGEST WHOLE NUMBER
NAMEABLE WITH 1,000 CHARACTERS
OF ENGLISH TEXT.



Bertrand Russell, 1872 – 1970

“The Berry Paradox”

G. G. Berry, 1867 – 1928

References / Acknowledgements / Further reading

- Scott Aaronson, *Who can name the bigger number?* www.scottaaronson.com/writings/bignumbers.html

- Archimedes, *The Sand Reckoner*

- Will Dison and Timothy Riley, *Hydra groups*, front.math.ucdavis.edu/1002.1945

- Martin Gardener, *Mathematical Games*, Scientific American, November 1977

- Wikipedia – especially the articles on *Ramsey Theory, Ramsey's Theorem, Graham's number* and *Knuth's arrow notation.*

Slides available at: www.math.cornell.edu/~riley/talks.html